# Robust control of distributed parameter mechanical systems using a multidimensional systems approach

B. CICHY[1], P. AUGUSTA[2], E. ROGERS[3], K. GAŁKOWSKI[1*], and Z. HURÁK[4]

[1] Institute of Control and Computation Engineering, University of Zielona Góra, 50 Podgórna St., 65-246 Zielona Góra, Poland
[2] Institute of Information Theory and Automation, Czech Academy of Sciences, Pod Vodarenskou vezi 4, 182 08 Prague, Czech Republic
[3] School of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, UK
[4] Department of Control Engineering, Czech Technical University in Prague, Karlovo nam. 13, 121 35 Prague, Czech Republic

**Abstract.** The unique characteristic of a repetitive processes is a series of sweeps, termed passes, through a set of dynamics defined over a finite duration. On each pass an output, termed the pass profile is produced which acts as on forcing function, and hence contributes to, the dynamics of the next pass profile. This leads to the possibility that the output, i.e. the sequence of pass profiles, will contain oscillations that increase in amplitude in the pass-to-pass direction. Such behavior cannot be controlled by application of standard linear systems control laws and instead they must be treated as two-dimensional (2D) systems where information propagation in two independent directions, termed pass-to-pass and along the pass respectively, is the defining feature. Physical examples of such processes include long-wall coal cutting and metal rolling. In this paper, stability analysis and control law design algorithms are developed for discrete linear repetitive processes where a plane, or rectangle, of information is propagated in the pass-to-pass direction. The possible use of such a model in the control of distributed parameter systems has been investigated in previous work and this paper considers an extension to allow for uncertainty in the model description.

**Key words:** robust control, distributed parameter mechanical systems, multidimensional systems.

## 1. Introduction

Multidimensional, or $n$D systems propagate information in $n > 1$ independent directions and arise in many areas, in particular, circuits, and image/signal processing. In the case of linear dynamics, this means that a transfer-function description is a function of $n$ indeterminates and this alone is a source of difficulty in terms of onward systems related analysis. For example, in the case of functions of more than one indeterminate the fundamental tool of primeness which is at the heart of the polynomial/transfer-function approach to many standard, termed 1D here, linear systems analysis and control law design problems is no longer a single concept and hence a direct extension of tools from this other area is, in general, not possible.

The case of discrete linear systems recursive in the upper right quadrant $(i, j)$: $i \geq 0$, $j \geq 0$ (where $i$ and $j$ denote the directions of information propagation) of the 2D plane has been the subject of much research effort over the years using, in the main, the Roesser [1] and Fornasini Marchesini [2] state-space models. More recently, productive research has been reported on robust control using a variety of approaches, see, for example, [3] and [4]. This paper considers discrete linear repetitive processes that are recursive in upper right quadrant of the 2D plane where information in one of the two independent directions only occurs over a finite duration.

The unique characteristic of a repetitive process (also termed a multipass process in the early literature) can be illustrated by considering machining operations where the material or workpiece involved is processed by a series of sweeps, or passes, of the processing tool. Assuming the pass length $\alpha < +\infty$ to be constant, the output vector, or pass profile, $y_k(p)$, $p = 0, 1, \ldots, (\alpha - 1)$, ($p$ being the independent spatial or temporal variable), generated on pass $k$ acts as a forcing function on, and hence contributes to, the dynamics of the new pass profile $y_{k+1}(p)$, $p = 0, 1, \ldots, (\alpha - 1)$, $k = 0, 1, \ldots$. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction, i.e. in the collection of pass profile vectors $\{y_k\}$.

Physical examples of repetitive processes include longwall coal cutting and metal rolling (for background on this and other physical examples see the references given in [5]). Also there are so-called algorithmic examples where adopting a repetitive process setting for analysis has clear advantages over alternative approaches to systems related analysis. These include iterative learning control schemes, e.g. [6] and iterative solution algorithms for dynamic nonlinear optimal control problems based on the maximum principle, e.g. [7]. In the case of iterative learning control for the linear dynamics case, it has recently been shown that the repetitive process setting can be used to design iterative learning control algorithms that have been experimentally verified on a gantry robot [8, 9] where, in particular, this design method, unlike alternatives, allows consideration of two possibly conflicting performance objectives to be included in the design process.

The links between systems described by partial differential equations and $n$D systems is an active area of research using, for example, the behavioral setting [10, 11] and this paper

*e-mail: k.galkowski@issi.uz.zgora.pl

also considers this general topic. Moreover, as with $n$D systems, there are a wide range of models for repetitive processes depending on the assumptions made concerning the contribution of the previous pass profile to the current one, i.e. the pass-to-pass updating structure. For example, it is possible to write down a model where it is a plane of information that is propagated in the pass-to-pass direction and previous work has shown that this can be used to model the discretized dynamics of certain classes of partial differential equations. In this paper the subject is the extension of this previous work to allow for uncertainty in the model structure and we begin in the next section with a summary of the essential background results.

The analysis in this paper will make extensive use of the well known Schur's complement formula for matrices and the elimination Lemma.

**Lemma 1.** [3] For any appropriately dimensioned matrices $\Sigma_1$, $\Sigma_2$, $F$ such that $F^T F \leq I$ and a scalar $\mu > 0$ the following holds

$$\Sigma_1 F \Sigma_2 + \Sigma_2^T F^T \Sigma_1^T \leq \mu^{-1} \Sigma_1 \Sigma_1^T + \mu \Sigma_2^T \Sigma_2. \tag{1}$$

Also $M > 0$, $M \geq 0$ (respectively $M < 0$, $M \leq 0$) is used to denote a real symmetric positive or positive semi-definite (respectively negative or semi-negative) definite matrix. Finally, the null and identity matrices of appropriate dimensions are denoted by $0$ and $I$ respectively.

## 2. Background

Consider the case of discrete dynamics along the pass and let $\alpha < \infty$ denote the pass length and $k \geq 0$ the pass number or index. Then discrete linear repetitive processes evolve over the subset of the positive quadrant in the 2D plane defined by $\{(p, k) \colon 0 \leq p \leq \alpha - 1, k \geq 0\}$, and their most basic state-space model [5] has the following form

$$x_{k+1}(p + 1) = A x_{k+1}(p) + B u_{k+1}(p) + B_0 y_k(p),$$

$$y_{k+1}(p) = C x_{k+1}(p) + D u_{k+1}(p) + D_0 y_k(p). \tag{2}$$

Here on the pass $k$, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, and $u_k(p) \in \mathbb{R}^r$ is the vector of control inputs.

In order to complete the process description it is necessary to specify the boundary conditions, that is, the pass state initial vector sequence and the initial pass profile. The simplest form of these is

$$x_{k+1}(0) = d_{k+1}, \qquad k \geq 0,$$

$$y_0(p) = f(p), \qquad 0 \leq p \leq \alpha - 1, \tag{3}$$

where the $n \times 1$ vector $d_{k+1}$ has known constant entries and $f(p)$ is an $m \times 1$ vector whose entries are known functions of $p$.

The stability theory [5] for linear repetitive processes is based on an abstract model in a Banach space setting which includes a large number of such processes as special cases. In this setting, a bounded linear operator mapping a Banach space into itself describes the contribution of the previous pass dynamics to the current one and the stability conditions are described in terms of properties of this operator. Noting again the unique feature of these processes, that is, oscillations that increase in amplitude from pass-to-pass (the $k$ direction in the notation for variables used so far in this paper), this theory is based on ensuring that such a response cannot occur by demanding that the output sequence of pass profiles generated $\{y_k\}$ has a bounded input bounded output stability property defined in terms of the norm on the underlying Banach space.

Two distinct forms of stability can be defined in this setting which are termed asymptotic stability and stability along the pass respectively. The former requires this bounded-input bounded-output property with respect to the, finite and fixed, pass length and the latter uniformly, that is, independent of the pass length. Asymptotic stability guarantees the existence of a so-called limit profile defined as the strong limit as $k \to \infty$ of the sequence $\{y_k\}_k$ and for processes described by (2) and (3) this is described by a standard, or 1D, discrete linear systems state-space model with state matrix $A_{lp} \colon = A + B_0(I - D_0)^{-1}C$. Hence it is possible for asymptotic stability to result in a limit profile which is unstable as a 1D discrete linear system, for example, $A = -0.5$, $B = 0$, $B_0 = 0.5 + \beta$, $C = 1$, $D = 0$, $D_0 = 0$, where $\beta$ is a real scalar satisfying $|\beta| \geq 1$. Stability along the pass prevents this from happening by demanding that the stability property be independent of the pass length, which can be analyzed mathematically by letting $\alpha \to \infty$.

The model (2) assumes the simplest possible pass-to-pass updating structure where at any point on the current pass the only previous pass contribution to both the state and pass profile updating arises from the same point on the previous pass. There are, however, physical examples where this is too simplistic, for example, in the long-wall coal cutting the previous pass profile is the height of the coal/stone interface above some datum and the cutting machine rests on this profile during the production of the next one. Hence it is not realistic to assume that at each point of the current pass the only contribution from the previous pass is from the same point.

There are many possible models that can be used to describe the previous pass profile contribution in repetitive processes and here the discrete linear repetitive processes considered are described by the following state-space model

$$x_{k+1}(l, m) = \sum_{i=-\epsilon}^{\epsilon} \sum_{j=-\varepsilon}^{\varepsilon} \tag{4}$$
$$\left( A^{i,j} x_k(l + i, m + j) + B^{i,j} u_k(l + i, m + j) \right),$$

where on pass $k$ $x_k(l, m) \in \mathbb{R}^n$ is the state vector, $u_k(l, m) \in \mathbb{R}^q$ is the control input vector, and $\epsilon > 0$ and $\varepsilon > 0$ are positive integers. The boundary conditions are

$$x_k(l, m) = 0, \quad -\epsilon \leq l < 0,\ 0 \leq m \leq \beta,\ k \geq 0,$$
$$x_k(l, m) = 0, \quad -\varepsilon \leq m < 0,\ 0 \leq l \leq \alpha,\ k \geq 0,$$
$$x_0(l, m) \doteq d_0(l, m), \quad 0 \leq l \leq \alpha,\ 0 \leq m \leq \beta,$$
$$x_k(\alpha - i, m) \doteq d_k(i, m), \quad 0 \leq m \leq \beta,\ 0 \leq i < \epsilon,\ k \geq 0,$$
$$x_k(l, \beta - j) \doteq d_k(l, j), \quad 0 \leq l \leq \alpha,\ 0 \leq j < \varepsilon,\ k \geq 0. \tag{5}$$

Here the process dynamics are defined over a finite fixed rectangle, i.e. $0 \leq l \leq \alpha - \epsilon, 0 \leq m \leq \beta - \varepsilon$ but at any point on pass $k + 1$ it is only the points in the so-called mask $-\epsilon \leq l \leq \epsilon$, $-\varepsilon \leq m \leq \varepsilon$, on the previous pass that contribute to the pass profile. The updating structure for the case when $\epsilon = \varepsilon = 1$ is illustrated in Fig. 1.
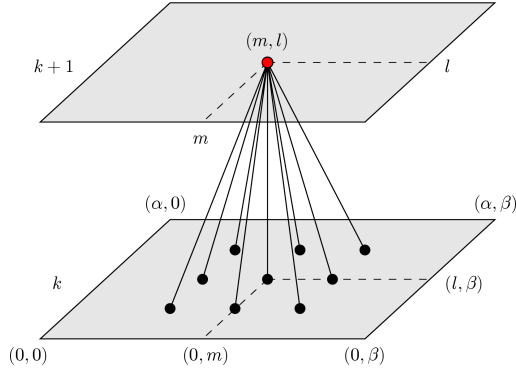


Fig. 1. Updating structure

In these processes it is a plane, or rectangle, of information which is propagated in the pass-to-pass direction. Note also that they share many joint features with the so-called spatially interconnected systems, which have already found numerous important physical applications, see, for example, [12] and references therein. This arises from the fact that some of the state-space models in this area can be rewritten as a discrete linear repetitive process state-space model (or its differential equivalent).

In the next section it is shown how the repetitive process model introduced in this section arises in the modeling of distributed parameter systems, using as an example the displacement of a flexible plate under the application of a force.

## 3. Distributed parameter dynamics modeled as a repetitive process

Consider Fig. 2 which shows a flexible plate to which a transverse external force is applied [13]. Suppose also that the resulting deformation dynamics can be modeled using a partial differential equation (PDE) of the following form first obtained by Lagrange in $1811$ (see, for example, [14] for full details)

$$
\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2 \frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4}
$$
$$
+ \frac{\rho}{D} \frac{\partial^2 w(x,y,t)}{\partial t^2} = \frac{q(x,y,t)}{D}, \tag{6}
$$

where $w$ is the lateral deflection in the $z$ direction [m], $\rho$ is the mass density per unit area [kg/m$^2$], $q$ is the transverse external force, with dimension of force per unit area [N/m$^2$], $\frac{\partial^2 w}{\partial t^2}$ is the acceleration in the $z$ direction [m/s$^2$], $D = E h^3 / (12 (1 - \nu^2))$, $\nu$ is Poisson's ratio, $h$ is thickness of the plate [m], and $E$ is Young's Modulus [N/m$^2$].
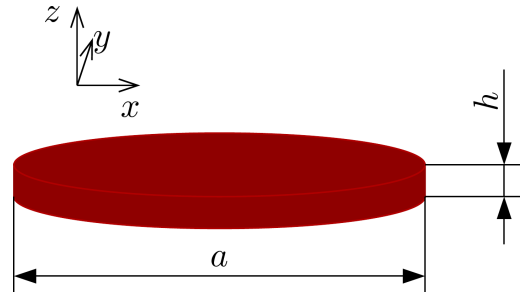


Fig. 2. Thin circular plate

If control action is to be applied, then often this will be implemented digitally and hence (6) must be discretized with respect to time. Moreover, if an array of actuators and zonal type wavefront sensors are to be used, discretization in the spatial variables is also required.

Finite difference (FD) methods are a well established numerical tool for solving PDEs (see, for instance, [15]). The basic principle of these methods is to cover the region where a solution is sought by a regular grid and to replace derivatives by differences using only values at these nodal points. There are many types of grids which can be used, e.g., rectangular, hexagonal, triangular or polar. Of these, the rectangular one is very appealing because of the very simple difference formulas which result. However, triangular or hexagonal grids are better fitted to the circular aperture and here we will consider a circular thin flexible plate and a triangular grid and derive the corresponding difference formulas and partial recurrence equation approximating the PDE (6). This results in the following recurrence approximating the dynamics

$$
\begin{aligned}
w_{l,m,k+1} = -\frac{D \, \Delta t^2}{\rho} \Big[ & P\, w_{l,m,k} + Q\left( w_{l-1,m-1,k} \right. \\
& \left. + w_{l-1,m+1,k} + w_{l+1,m-1,k} + w_{l+1,m+1,k} \right) \\
& + R\left( w_{l-2,m,k} + w_{l+2,m,k} \right) \\
& + S\left( w_{l,m-2,k} + w_{l,m+2,k} \right) \Big] + 2\, w_{l,m,k} \\
& - w_{l,m,k-1} + \frac{\Delta t^2}{\rho}\, q_{l,m,k},
\end{aligned} \tag{7}
$$

where

$$
P = \frac{6}{\Delta x^4} + \frac{8}{\Delta x^2 \, \Delta y^2} + \frac{6}{\Delta y^4}
$$

$$
Q = -\frac{2}{\Delta x^4} - \frac{2}{\Delta x^2 \, \Delta y^2} - \frac{2}{\Delta y^4}
$$

$$
R = \frac{1}{\Delta x^4},
$$

$$
S = \frac{1}{\Delta y^4},
$$

where $\Delta t, \Delta x, \Delta y$ are respectively the time and space differences. The detailed derivation of this model can be found in [16].

Before proceeding, it is essential to verify if the model obtained is an acceptably accurate approximation to the original

dynamics given by the PDE. This is by means of a stability analysis of the iterative FD scheme, the objective being to determine whether the iterative scheme given by (7) converges to a solution. In particular, we determine a relationship between $\Delta t$ and $\Delta x$ which guarantees convergence by applying von Neumann analysis (a standard technique in this general area). This yields the following requirement (for the details again see [16]).

$$\Delta t \leq \frac{3\sqrt{3\,\rho}\,\Delta x^2}{\sqrt{136\,D - 9\,\rho\,\Delta x^4}}. \tag{8}$$

## 4. Stability analysis and control law design

It has been shown in previous work [8, 9] that stability along the pass theory for discrete linear repetitive processes described by (2) can be used to design ILC algorithms for finite-dimensional discrete linear systems and that some of these have been experimentally verified. This previous analysis used a Lyapunov function characterization of stability along the pass, where the most common form of this function is

$$V(k,p) = x_{k+1}^T(p)Qx_{k+1}(p) + y_k(p)Wy_k(p), \tag{9}$$

with $Q \succ 0$ and $W \succ 0$, that is, the sum of quadratic terms in the current pass state and previous pass profiles respectively for given $k$ and $p$.

In the case of (4), a candidate, so-called 'local' Lyapunov function for given $k$, $l$, $m$

$$V_k(l,m) \doteq \sum_{i=-\epsilon}^{\epsilon} \sum_{j=-\varepsilon}^{\varepsilon} x_k^T(l+i,m+j)V^{i,j}x_k(l+i,m+j), \tag{10}$$

where $V^{i,j} > 0$, $i = -\epsilon, \ldots, \epsilon$, $j = -\varepsilon, \ldots, \varepsilon$. This function is the local energy for the considered mask (i.e. $-\epsilon \leq l \leq \epsilon$, $-\varepsilon \leq m \leq \varepsilon$). The so-called total Lyapunov function is given by

$$\mathcal{V}_k \doteq \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} x_k^T(i,j)Vx_k(i,j), \tag{11}$$

where

$$V \doteq \sum_{i=-\epsilon}^{\epsilon} \sum_{j=-\varepsilon}^{\varepsilon} V^{i,j}. \tag{12}$$

The associated increment for the local Lyapunov function is defined as

$$\Delta V_k(l,m) \doteq x_{k+1}^T(l,m)Vx_{k+1}(l,m)$$
$$- \sum_{i=-\epsilon}^{\epsilon} \sum_{i=-\varepsilon}^{\varepsilon} x_k^T(l+i,m+j)V^{i,j}x_k(l+i,m+j). \tag{13}$$

Motivated by physical arguments that the total energy at the pass (finite for all of them) should decrease from pass-to-pass we introduce the following total Lyapunov function increment

$$\Delta \mathcal{V}_k \doteq \mathcal{V}_{k+1} - \mathcal{V}_k. \tag{14}$$

The increment (14) has the same structure as that for (9). Moreover, it has been shown elsewhere [5] that stability along the pass of processes described by (2) holds when the increment of the Lyapunov function (9) is negative definite for all

possible values of $\alpha$ and $k$. It is also straightforward to argue that this stability theory extends to processes for which (11) is a candidate Lyapunov function. Hence the proof of the following result is omitted here.

**Theorem 1.** A discrete linear repetitive process described by (4) and (5) is stable over $R = \{(k,l,m)\colon k = 0,1,\ldots,N; l = 0,1,\ldots,\alpha-1, m = 0,1,\ldots,\beta-1\}$ for any choice of the positive integers $N$ and $\alpha > 1$, $\beta > 1$ if

$$\Delta \mathcal{V}_k < 0$$

$\forall x_{k+1}(l,m)$, $l = 0,1,\ldots,\alpha-1$, $m = 0,1,\ldots,\beta-1$, $0 \leq k \leq N$.

This result can also be expressed in terms of a Linear Matrix Inequality (LMI), which provides a computational test for this property, [16].

**Theorem 2.** A discrete linear repetitive process described by (4) and (5) is stable along the pass if there exist $V^{i,j} > 0$, $\forall i \in \{-\epsilon,\ldots,0,\ldots,\epsilon\}$, $\forall j \in \{-\varepsilon,\ldots,0,\ldots,\varepsilon\}$ such that the following LMI holds

$$\mathbf{A}^T\mathbf{V}\mathbf{A} - \mathbf{V} < 0, \tag{15}$$

where

$$\mathbf{V} \doteq \bigoplus_{i=-\epsilon}^{\epsilon} \bigoplus_{j=-\varepsilon}^{\varepsilon} V^{i,j} \tag{16}$$

and $\bigoplus$ denotes the direct sum of matrices, i.e. for two matrices say $X_1$ and $X_2$

$$X_1 \bigoplus X_2 = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

and

$$\mathbf{A} \doteq \begin{bmatrix} A^{-\epsilon,-\varepsilon} & \cdots & A^{-\epsilon,\varepsilon} \\ \vdots & \ddots & \vdots \\ A^{\epsilon,-\varepsilon} & \cdots & A^{\epsilon,\varepsilon} \end{bmatrix}. \tag{17}$$

The following result is a more computationally feasible method for testing stability and extends directly to control law design (Theorem 4).

**Theorem 3.** A discrete linear repetitive process described by (4) and (5) is stable along the pass if there exist matrices $\mathbf{V} > 0$ (defined in (16)) and $\mathbf{G}$ such that

$$\begin{bmatrix} -\mathbf{V} & \mathbf{A}\mathbf{G} \\ \mathbf{G}^T\mathbf{A}^T & -\mathbf{G} - \mathbf{G}^T + \mathbf{V} \end{bmatrix} < 0, \tag{18}$$

where

$$\mathbf{G} \doteq \bigoplus_{i=-\epsilon}^{\epsilon} \bigoplus_{j=-\varepsilon}^{\varepsilon} G^{i,j}. \tag{19}$$

Suppose now that a control law of the following form is applied to a process described by (4) and (5)

$$
\begin{bmatrix} u_k^{-\epsilon,-\varepsilon}(l,m) \\ \vdots \\ u_k^{0,0}(l,m) \\ \vdots \\ u_k^{\epsilon,\varepsilon}(l,m) \end{bmatrix} = \mathbf{K} \begin{bmatrix} x_k(l-\epsilon,m-\varepsilon) \\ \vdots \\ x_k(l,m) \\ \vdots \\ x_k(l+\epsilon,m+\varepsilon) \end{bmatrix}, \qquad (20)
$$

where

$$
\mathbf{K} \doteq \bigoplus_{i=-\epsilon}^{\epsilon} \bigoplus_{j=-\varepsilon}^{\varepsilon} K^{i,j}. \qquad (21)
$$

Then interpreting (15) in terms of the resulting state-space model of the controlled process gives the following sufficient condition for stability along the pass

$$
(\mathbf{A} + \mathbf{BK})^T \mathbf{V} (\mathbf{A} + \mathbf{BK}) - \mathbf{V} < 0, \qquad (22)
$$

where the matrix $\mathbf{B}$ is given by

$$
\mathbf{B} \doteq \begin{bmatrix} B^{-\epsilon,-\varepsilon} & \cdots & B^{-\epsilon,\varepsilon} \\ \vdots & \ddots & \vdots \\ B^{\epsilon,-\varepsilon} & \cdots & B^{\epsilon,\varepsilon} \end{bmatrix}. \qquad (23)
$$

For control law design, the following result [16] can be used.

**Theorem 4.** Suppose that a control law of the form (20) is applied to a discrete linear repetitive process described by (4) and (5). Then the resulting controlled process is stable along the pass if there exists a block diagonal matrix $\mathbf{V} > 0$ (defined in (16)), a matrix $\mathbf{G}$ (defined in (19)) and a matrix $\mathbf{N}$ such that

$$
\begin{bmatrix} -\mathbf{V} & \mathbf{AG} + \mathbf{BN} \\ (\mathbf{AG} + \mathbf{BN})^T & -\mathbf{G} - \mathbf{G}^T + \mathbf{V} \end{bmatrix} < 0, \qquad (24)
$$

where

$$
\mathbf{N} \doteq \bigoplus_{i=-\epsilon}^{\epsilon} \bigoplus_{j=-\varepsilon}^{\varepsilon} N^{i,j} \qquad (25)
$$

and the matrices $\mathbf{A}$ and $\mathbf{B}$ are defined by (17) and (23) respectively. Also if (24) holds, a stabilizing $\mathbf{K}$ in the control law (20) is given by

$$
\mathbf{K} = \mathbf{NG}^{-1}. \qquad (26)
$$

## 5. Robustness

In this section we consider the case when there is uncertainty associated with the process state-space model (4). One way of addressing this is to assume that the uncertainty can be modeled as additive perturbations to the block matrices $\mathbf{A}$ of (17) and $\mathbf{B}$ of (23) of the following norm bounded form

$$
\begin{aligned} A_p &= \mathbf{A} + \widehat{\Delta \mathbf{A}}, \\ B_p &= \mathbf{B} + \widehat{\Delta \mathbf{B}}, \end{aligned} \qquad (27)
$$

where

$$
\begin{aligned} \widehat{\Delta \mathbf{A}} &:= \begin{bmatrix} \Delta A^{-\epsilon,-\varepsilon} & \cdots & \Delta A^{-\epsilon,\varepsilon} \\ \vdots & \ddots & \vdots \\ \Delta A^{\epsilon,-\varepsilon} & \cdots & \Delta A^{\epsilon,\varepsilon} \end{bmatrix}, \\ \\ \widehat{\Delta \mathbf{B}} &:= \begin{bmatrix} \Delta B^{-\epsilon,-\varepsilon} & \cdots & \Delta B^{-\epsilon,\varepsilon} \\ \vdots & \ddots & \vdots \\ \Delta B^{\epsilon,-\varepsilon} & \cdots & \Delta B^{\epsilon,\varepsilon} \end{bmatrix}, \end{aligned} \qquad (28)
$$

with

$$
\Delta A^{i,j} = H^{i,j} F E_1^{i,j},
$$
$$
\Delta B^{i,j} = H^{i,j} F E_2^{i,j}
$$

$\forall i,j \in (-\epsilon,-\varepsilon),\dots,(0,0),\dots,(\epsilon,\varepsilon)$. Also the matrix $F$ is required to satisfy

$$
F^T F \leq I. \qquad (29)
$$

Obviously, this is a form of constrained uncertainty and therefore may be somewhat restrictive as not all possible types can be modeled in this way. However, as demonstrated by the numerical example given in the next section, the resulting algorithms can be effectively applied to the application area considered in this work.

Introduce the following notation

$$
\begin{bmatrix} \widehat{\Delta \mathbf{A}} & \widehat{\Delta \mathbf{B}} \end{bmatrix} = \mathbf{H} F \mathbf{E}, \qquad (30)
$$

where

$$
\mathbf{E} = \begin{bmatrix} \mathbf{E_1} & \mathbf{E_2} \end{bmatrix},
$$
$$
\mathbf{E_1} = \begin{bmatrix} E_1^{i,j} & E_1^{i,j} & \cdots & E_1^{i,j} \end{bmatrix},
$$
$$
\mathbf{E_2} = \begin{bmatrix} E_2^{i,j} & E_2^{i,j} & \cdots & E_2^{i,j} \end{bmatrix},
$$
$$
\mathbf{H} = \begin{bmatrix} (H^{i,j})^T & (H^{i,j})^T & \cdots & (H^{i,j})^T \end{bmatrix}^T,
$$

$\forall i,j \in (-\epsilon,-\varepsilon),\dots,(0,0),\dots,(\epsilon,\varepsilon)$. Then we can apply Theorem 3 to conclude that stability along the pass holds in this case provided there exist matrices $\mathbf{V} > 0$ and $\mathbf{G}$ (of the form (16) and (19) respectively) such that

$$
\begin{bmatrix} -\mathbf{V} & (\mathbf{A} + \mathbf{H}F\mathbf{E_1})\mathbf{G} \\ \mathbf{G}^T(\mathbf{A} + \mathbf{H}F\mathbf{E_1})^T & -\mathbf{G} - \mathbf{G}^T + \mathbf{V} \end{bmatrix} < 0. \qquad (31)
$$

The difficulty with this last condition is that the matrix $F$ has unknown entries and hence it is not applicable as a computable stability test. To remove this, we have the following result as an obvious consequence Lemma 1.

**Lemma 2.** There exist matrices $\mathbf{V} > 0$ (defined in (16)) and $\mathbf{G}$ (defined in (19)) such that (31) holds for all $F$ satisfying (29) if, and only if, there exists a real scalar $\mu > 0$ such that

$$\begin{bmatrix} \mu\mathbf{G}^T\mathbf{E_1}^T\mathbf{E_1}\mathbf{G} - \mathbf{V} & \mathbf{AG} \\ \mathbf{G}^T\mathbf{A}^T & \mu^{-1}\mathbf{HH}^T - \mathbf{G} - \mathbf{G}^T + \mathbf{V} \end{bmatrix} < 0. \tag{32}$$

Now we have the following computable stability condition.

**Theorem 5.** A discrete linear repetitive processes described by (4) and (5) with uncertainty of the form defined by (27)–(29) is stable along the pass if there exist matrices $\mathbf{V} > 0$ (defined in (16)), $\mathbf{G}$ (defined in (19)) and a real scalar $\eta > 0$ such that the following LMI holds

$$\begin{bmatrix} -\mathbf{V} & \mathbf{AG} & \mathbf{G}^T\mathbf{E_1}^T & 0 \\ \mathbf{G}^T\mathbf{A}^T & -\mathbf{G} - \mathbf{G}^T + \mathbf{V} & 0 & \eta\mathbf{H} \\ \mathbf{E_1}\mathbf{G} & 0 & -\eta I & 0 \\ 0 & \eta\mathbf{H}^T & 0 & \eta I \end{bmatrix} < 0. \tag{33}$$

**Proof.** First, apply the Schur's complement formula to (32) and set $\eta = \mu^{-1}$ to obtain

$$\begin{bmatrix} -\mathbf{V} & \mathbf{AG} & \mathbf{G}^T\mathbf{E_1}^T \\ \mathbf{G}^T\mathbf{A}^T & \eta\mathbf{HH}^T - \mathbf{G} - \mathbf{G}^T + \mathbf{V} & 0 \\ \mathbf{E_1}\mathbf{G} & 0 & -\eta I \end{bmatrix} < 0.$$

Now apply the Schur's complement formula to this last result and then pre and post-multiply the result $\text{diag}\{I, I, I, \eta\}$ to obtain (33).

Suppose now that a control law of the form (20) is applied to this uncertain process. Then routine manipulations show that the resulting controlled process state-space model is of the form to which Theorem 4 can be applied. Again, however, this route cannot be used as an effective control law method since the matrix $F$ has unknown entries and also the condition which results is not of LMI form. The next result removes this difficulty.

**Theorem 6.** Suppose that a control law of the form (20) is applied to a discrete linear repetitive process described by (4) and (5) whose defining matrices have uncertainty associated with them of the form (27)–(29). Then the controlled process is stable along the pass if there exist matrices $\mathbf{V} > 0$ (defined in (16)), $\mathbf{N}$ (defined in (25)) and $\mathbf{G}$ (defined in (19)) and a real scalar $\eta > 0$ such that the following LMI holds

$$\begin{bmatrix} -\mathbf{V} & \star & \star & \star \\ (\mathbf{AG} + \mathbf{BN})^T & -\mathbf{G} - \mathbf{G}^T + \mathbf{V} & \star & \star \\ \mathbf{E_1}\mathbf{G} + \mathbf{E_2}\mathbf{N} & 0 & -\eta I & \star \\ 0 & \eta\mathbf{H}^T & 0 & -\eta I \end{bmatrix} < 0. \tag{34}$$

If (34) holds, then a stabilizing $\mathbf{K}$ in the control law (20) is given by

$$\mathbf{K} = \mathbf{NG}^{-1}, \tag{35}$$

with $\mathbf{K}$ defined in (21). The symbol $\star$ denotes symmetric block entries i.e. $(i, j) = (j, i)$.

**Proof.** First, apply the Schur's complement formula to the result of Lemma 2 interpreted for the controlled process here, set $\eta = \mu^{-1}$, apply Schur's formula to the result, and finally pre and post-multiply the outcome of this last step by $\text{diag}\{I, I, I, \eta\}$ to obtain

$$\begin{bmatrix} -\mathbf{V} & \star & \star & \star \\ \mathbf{G}^T(\mathbf{A}^T + \mathbf{K}^T\mathbf{B}^T) & -\mathbf{G} - \mathbf{G}^T + \mathbf{V} & \star & \star \\ (\mathbf{E_1} + \mathbf{E_2}\mathbf{K})\mathbf{G} & 0 & -\eta I & \star \\ 0 & \eta\mathbf{H}^T & 0 & -\eta I \end{bmatrix} < 0.$$

Setting $\mathbf{KG} = \mathbf{N}$ now completes the proof.

## 6. A numerical example

Consider the case when the plate parameters are given in Table 1 and the initial plate deflection is zero, that is, the forces and moments acting on the plate due to its weight are neglected and hence the initial condition is

$$w_{l,m,k}\Big|_{t=0} = 0.$$

Table 1
Plate parameters

| Parameter | Value |
|---|---|
| diameter $a$ | 1 m |
| thickness $h$ | $3.2004 \cdot 10^{-3}$ m |
| mass density per unit area $\rho$ | 2700 kg/m$^2$ |
| Young's Modulus $E$ | $7.11 \times 10^{10}$ m$^2$ |
| Poisson ratio $\nu$ | 0.3 |

Suppose also that the edge of the plate is clamped. Then the plate deflection and its derivative at the edge is always equal to zero, and the boundary conditions are

$$w(x, y, t)\Big|_{x, y \in \partial D} = 0,$$

$$\frac{\partial w(x, y, t)}{\partial x}\Big|_{x, y \in \partial D} = 0,$$

$$\frac{\partial w(x, y, t)}{\partial y}\Big|_{x, y \in \partial D} = 0,$$

where $\partial \mathcal{D}$ denotes the boundary of the region where a solution is sought. Also at every boundary point the following conditions must hold

$$w_{l,m,k} = 0,$$

$$w_{l-1,m-1,k} + w_{l-1,m+1,k} - w_{l+1,m-1,k} - w_{l+1,m+1,k} = 0,$$

(For the details again see (16).)

Consider now the application of Theorem 5 to this example. Then the corresponding LMI does not have a solution and hence we proceed to consider the design of a stabilizing control law using Theorem 6. In order to do this we must employ a mapping from the triangular grid used to approximate the process dynamics to the linear ordering used in Theorems 5 and 6. It is hence convenient to define the function $\varphi(\omega) \mapsto \{i, j\}$ which maps indices as follows

$$\varphi(1) \mapsto \{0, -2\}, \qquad \varphi(2) \mapsto \{-1, -1\},$$
$$\varphi(3) \mapsto \{1, -1\}, \qquad \varphi(4) \mapsto \{-2, 0\},$$
$$\varphi(5) \mapsto \{0, 0\}, \qquad \varphi(6) \mapsto \{2, 0\}, \qquad (36)$$
$$\varphi(7) \mapsto \{-1, 1\}, \qquad \varphi(8) \mapsto \{1, 1\},$$
$$\varphi(9) \mapsto \{0, 2\}$$

and additionally

$$\begin{aligned} \varphi(\omega, 1) &\mapsto i, \\ \varphi(\omega, 2) &\mapsto j. \end{aligned} \qquad (37)$$

For example, $\varphi(7, 1) \mapsto -1$ and $\varphi(7, 2) \mapsto 1$. Then we have

$$\mathbf{A} \doteq \begin{bmatrix} A^{\varphi(1)} & \cdots & A^{\varphi(9)} \\ \vdots & \ddots & \vdots \\ A^{\varphi(1)} & \cdots & A^{\varphi(9)} \end{bmatrix}, \qquad (38)$$

$$\mathbf{B} \doteq \begin{bmatrix} B^{\varphi(1)} & \cdots & B^{\varphi(9)} \\ \vdots & \ddots & \vdots \\ B^{\varphi(1)} & \cdots & B^{\varphi(9)} \end{bmatrix}, \qquad (39)$$

where the $2 \times 2$ matrices $A^{\varphi(\omega)}$, $B^{\varphi(\omega)}$ and $\omega = 1, 2, \ldots, 9$, are constructed from the appropriate coefficients of the underlying discrete equation as

$$A^{\varphi(1)} = A^{\varphi(9)} = \begin{bmatrix} -\dfrac{D\,\Delta t^2}{\rho} S & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -6.4219 \cdot 10^{-4} & 0 \\ 0 & 0 \end{bmatrix},$$

$$A^{\varphi(2)} = A^{\varphi(3)} = A^{\varphi(7)} = A^{\varphi(8)} = \begin{bmatrix} -\dfrac{D\,\Delta t^2}{\rho} Q & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1.6697 \cdot 10^{-2} & 0 \\ 0 & 0 \end{bmatrix},$$

$$A^{\varphi(5)} = \begin{bmatrix} -\dfrac{D\,\Delta t^2}{\rho} P + 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1.9461 & -1 \\ 1 & 0 \end{bmatrix},$$

$$A^{\varphi(4)} = A^{\varphi(6)} = \begin{bmatrix} -\dfrac{D\,\Delta t^2}{\rho} R & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -5.7798 \cdot 10^{-3} & 0 \\ 0 & 0 \end{bmatrix},$$

$$B^{\varphi(1)} = B^{\varphi(2)} = B^{\varphi(3)} = B^{\varphi(4)} = B^{\varphi(5)} = B^{\varphi(6)} = B^{\varphi(7)}$$
$$= B^{\varphi(8)} = B^{\varphi(9)} = \begin{bmatrix} \dfrac{\Delta t^2}{\rho} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3.7037 \cdot 10^{-12} & 0 \\ 0 & 0 \end{bmatrix}.$$

Now suppose that a control law of the following form is applied to this example

$$\begin{bmatrix} u_k^{\varphi(1)}(l, m) \\ \vdots \\ u_k^{\varphi(9)}(l, m) \end{bmatrix} = \mathbf{K} \begin{bmatrix} x_k\big(l + \varphi(1, 1), m + \varphi(1, 2)\big) \\ \vdots \\ x_k\big(l + \varphi(9, 1), m + \varphi(9, 2)\big) \end{bmatrix}, \quad (40)$$

where

$$\mathbf{K} \doteq \bigoplus_{\omega=1}^{9} K^{\varphi(\omega)}, \qquad (41)$$

when there is uncertainty in the values of the parameters $a$, $h$, and $\rho$ (showed in Table 1) of the form

$$a = a \pm \Delta a, \qquad \Delta a = 0.05a,$$
$$h = h \pm \Delta h, \qquad \Delta h = 0.05h,$$
$$\rho = \rho \pm \Delta \rho, \qquad \Delta \rho = 0.1\rho.$$

Then the matrices $\mathbf{H}$ and $\mathbf{E}$ in this particular case are

$$\mathbf{E_1} = \begin{bmatrix} E_1 & E_2 & E_2 & E_2 & E_2 & E_3 \\ \mathbf{0}_{17 \times 2} & \mathbf{0}_{17 \times 2} & \mathbf{0}_{17 \times 2} & \mathbf{0}_{17 \times 2} & \mathbf{0}_{17 \times 2} & \mathbf{0}_{17 \times 2} \end{bmatrix}$$
$$\begin{bmatrix} & E_3 & E_4 & E_4 \\ & \mathbf{0}_{17 \times 2} & \mathbf{0}_{17 \times 2} & \mathbf{0}_{17 \times 2} \end{bmatrix},$$

$$\mathbf{E_2} = \mathbf{0}_{18 \times 18}$$

and

$$\mathbf{H} = \begin{bmatrix} H_1 & H_5 & H_8 & H_8 & H_8 & H_8 & H_8 & H_8 & H_8 \\ H_2 & H_3 & H_9 & H_9 & H_9 & H_9 & H_9 & H_9 & H_9 \\ H_2 & H_4 & H_7 & H_9 & H_6 & H_6 & H_6 & H_6 & H_6 \\ H_2 & H_4 & H_6 & H_7 & H_6 & H_6 & H_6 & H_6 & H_6 \\ H_2 & H_4 & H_6 & H_6 & H_7 & H_6 & H_6 & H_6 & H_6 \\ H_2 & H_4 & H_6 & H_6 & H_6 & H_7 & H_6 & H_6 & H_6 \\ H_2 & H_4 & H_6 & H_6 & H_6 & H_6 & H_7 & H_6 & H_6 \\ H_2 & H_4 & H_6 & H_6 & H_6 & H_6 & H_6 & H_7 & H_6 \\ H_2 & H_4 & H_6 & H_6 & H_6 & H_6 & H_6 & H_6 & H_7 \end{bmatrix}$$

where

$$E_1 = \begin{bmatrix} 0.0179 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} -5.5368 \cdot 10^{-3} & 0 \end{bmatrix},$$
$$E_3 = \begin{bmatrix} 1.9166 \cdot 10^{-3} & 0 \end{bmatrix}, \qquad E_4 = \begin{bmatrix} 2.1296 \cdot 10^{-4} & 0 \end{bmatrix},$$

and

$$H_1 = \begin{bmatrix} -0.3333 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} -0.3333 & 0 \\ 0 & 0 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} -0.1179 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H_4 = \begin{bmatrix} -0.1179 & 0 \\ 0 & 0 \end{bmatrix}$$

$$H_5 = \begin{bmatrix} 0.9428 & 0 \\ 0 & 0 \end{bmatrix}, \qquad H_6 = \begin{bmatrix} -0.0923 & 0 \\ 0 & 0 \end{bmatrix}$$

$$H_7 = \begin{bmatrix} 0.9077 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H_8 = \mathbf{0}_{2 \times 2}$$

$$H_9 = \begin{bmatrix} -0.3536 & 0 \\ 0 & 0 \end{bmatrix}.$$

Applying Theorem 6 now gives the stabilizing control law matrix $\mathbf{K}$ in this case as

$$K^{\varphi(1)} = \begin{bmatrix} -2.29 \cdot 10^{11} & 2.08 \cdot 10^{11} \\ 0 & 0 \end{bmatrix},$$

$$K^{\varphi(\omega)} = \begin{bmatrix} -2.5 \cdot 10^{-4} & 0 \\ 0 & 0 \end{bmatrix}, \quad \omega = 2, 3, \dots, 5,$$

$$K^{\varphi(\omega)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \omega = 6, 7, \dots, 9.$$

As a numerical example, consider the boundary conditions of Fig. 3. Then Figs. 4, 5 and 6 show response of the controlled response at nodes on the middle diagonal, the deflection at a node in the middle of the plate, and the control signal at the same node in the middle of the plate respectively. Figure 7 shows deflection of the complete plate after 5 ms have elapsed. These confirm that a stabilizing control law has been produced and since it is a regulator problem, the initial deflection is eventually returned to rest. The input signal in this example is very high and is clearly related to very small values of the entries in the matrix $B^{\varphi(\omega)}$, $\omega = 1, 2, \dots, 9$. Clearly such a signal cannot be actually implemented and further detailed design studies are clearly required to ensure that the control effort required is within the range of available actuators.
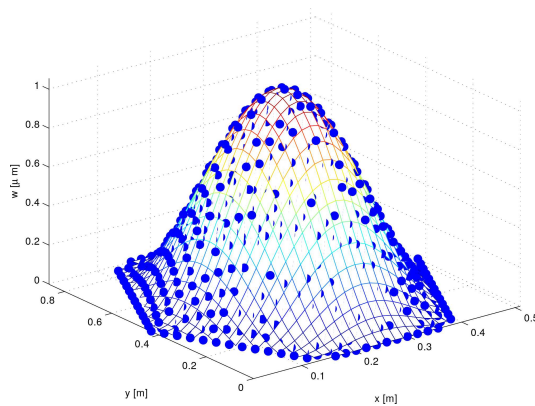
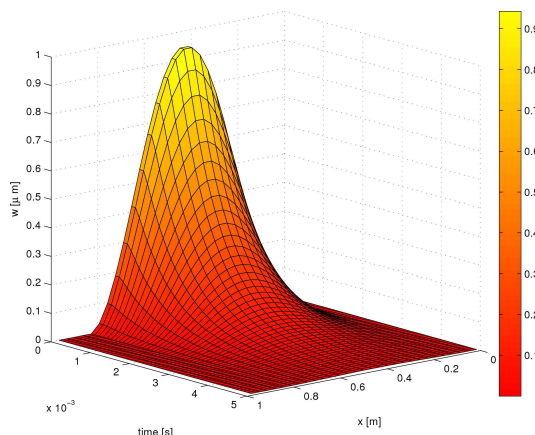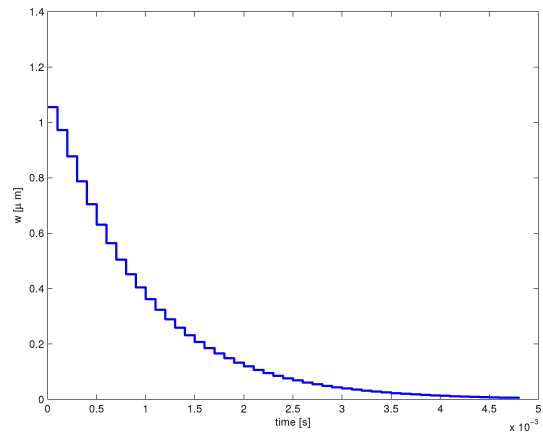Fig. 5. Deflection at a node in the middle of the plate, controlled process with control law (40)

Fig. 6. Control signal at node in the middle of mirror, controlled process with control law (40)

Fig. 3. Initial conditions

Fig. 4. Deflection at nodes on the middle diagonal, controlled system with control law (40) applied

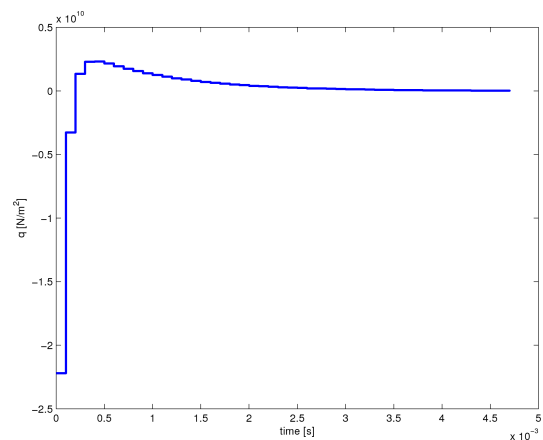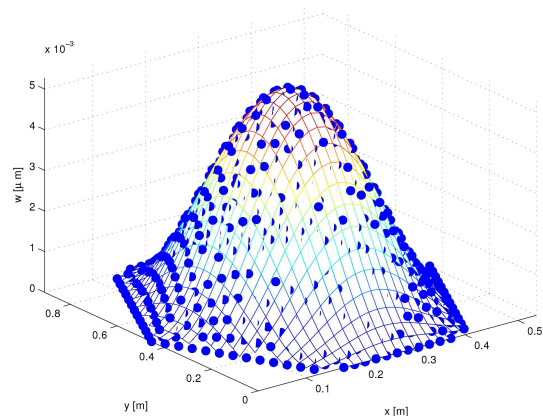Fig. 7. Deflection of the plate at time 5 ms, controlled process with control law (40)

## 7. Conclusions

This paper has produced the extensions to the robust case of the substantial results of [16] on a new model for repetitive processes where it is a plane of information which is propagated in the pass-to-pass direction. This makes the system three dimensional (3D) and motivation for considering such a model has been given by showing how it arises the discretization

of the dynamics of distributed parameter systems in the form of a fourth order partial differential equation governing the transverse vibrations of a thin plate.

Stability along the pass for this new repetitive process model has been defined in energy terms and it has been shown that the resulting condition can be expressed in terms of an LMI. Moreover, this also provides a basis on which to specify and design robust control laws for distributed parameter systems with, in particular, immediate recourse to well documented and powerful computational tools in the form of LMIs. The analysis here is based on sufficient but not necessary stability conditions and hence a degree of conservativeness could be present but experience in other repetitive process theory strongly suggests that this is often not very severe.

The results in this paper are among the first on the robust control of this form of repetitive process dynamics and much remains to be done both in terms of theory and (potential) applications. This is especially true given the emphasis now on distributed control for application to, for example, adaptive optics systems (see, for example, [17] for background) where [13] contains some results from analysis in an $n$D systems setting (this is based on polynomial methods and is hence limited in terms of cases to which design can be completed). Other potential application areas for a repetitive process based approach to the control of distributed parameter systems include scene based iterative learning control [18] and also diffusion control in irrigation applications [19]. Also, via the connection to iterative learning control, the repetitive process setting could also be used in repetitive control (for possibly relevant work see [20]).

Progress in this general area will only be feasible after much further research is completed. Obvious topics for this include i) the discretization methods possible since FE methods may often not be appropriate or even applicable and then the question to be answered is: can we again get to a repetitive process model approximation to the dynamics which is suitable and realistic basis for control law design, ii) the use of model validation tools beyond the classical von Neumann approach used here, iii) exactly what classes of partial differential equations can be treated in this way, iv) robust control design since we have always been using an approximate model for design and initial control law evaluation, and v) comparison (where applicable) with alternative approaches, such as those of [12].

REFERENCES

[1] R.P. Roesser, "A discrete state-space model for linear image processing", *IEEE Trans. on Automatic Control* 20, (1), 1–10 (1975).

[2] E. Fornasini and G. Marchesini, "Doubly indexed dynamical systems: state-space models and structural properities", *Theory of Computing Systems* 12, (1), 59–72 (1978).

[3] C. Du and L. Xie, "Stability analysis and stabilization of uncertain two-dimensional discrete systems: an LMI approach", *IEEE Trans. on Circuits and Systems I: Fundamental Theory and Applications* 46, 1371–1374 (1999).

[4] H.D. Tuan, P. Apkarian, and T.Q. Nguyen, "Robust mixed $H_2/H_\infty$ filtering of 2-D systems", *IEEE Trans. on Signal Processing* 50 (7), 1759–1771 (2002).

[5] E. Rogers, K. Gałkowski, and D.H. Owens, "Control systems theory and applications for linear repetitive processes", in: *Lecture Notes in Control and Information Sciences*, vol. 349, Springer-Verlag, Berlin, 2007.

[6] H-S. Ahn, Y. Chen, and K.L. Moore, "Iterative learning control: brief survey and categorization", *IEEE Trans. on Systems Man and Cybernetics* C 37 (6), 1109–1121 (2007).

[7] P.D. Roberts, "Numerical investigations of a stability theorem arising from 2-dimensional analysis of an iterative optimal control algorithm", *Multidimensional Systems and Signal Processing* 11 (1/2), 109–124 (2000).

[8] Ł. Hładowski, Z. Cai, K. Gałkowski, E. Rogers, C.T. Freeman, and P.L. Lewin, "Using 2D systems theory to design output signal based iterative learning control laws with experimental verification", *Proc. 47th IEEE Conf. on Decision and Control* 1, 3026–3031 (2008).

[9] Ł. Hładowski, K. Gałkowski, Z. Cai, E. Rogers, C.T. Freeman, and P.L. Lewin, "Experimentally supported 2D systems based iterative learning control law design for error convergence and performance", *Control Engineering Practice* 18 (4), 339–348 (2010).

[10] J. Wood, U. Oberst, E. Rogers, and D.H. Owens, "A behavioural approach to the pole structure of one-dimensional and multidimensional linear systems", *SIAM J. on Control and Optimization* 38,(2), 627–661 (2000).

[11] H. Pillai, J. Wood, and E. Rogers, "On the homomorphisms of n-D behaviors", *IEEE Trans. on Circuits and Systems Part I: Fundamental Theory and Applications* 49 (6), 732–742 (2002).

[12] G.E. Dullerud and R. D'Andrea, "Distributed control of heterogeneus systems", *IEEE Trans. Automatic Control* 49 (12), 2113–2128 (2004).

[13] P. Augusta and Z. Hurák, "Multidimensional transfer function model of a deformable mirror in adaptive optics systems", pp. 1556–1561, Int. Symposium on Mathematical Theory of Networks and Systems, Kyoto, 2006.

[14] S. Timoshenko and S. Woinowski-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York, 1959.

[15] J.C. Strikwerda, *Finite Difference Schemes and Partial Differential Equations*, Wadsworth and Brooks, London, 1989.

[16] B. Cichy, P. Augusta, E. Rogers, K. Gałkowski, and Z. Hurák, "On the control of distributed parameter systems using a multidimensional systems setting", *Mechanical Systems and Signal Processing* 22, 1566–1581 (2008).

[17] P.I. Belly, *The Design and Construction of Large Optical Telescopes*, Springer-Verlag, Berlin, 2003.

[18] K.L. Moore, M. Ghosh, and Y. Chen, "Spatial-based iterative learning control for motion control applications", *Meccanica* 42 (2), 167–175 (2007).

[19] K.L. Moore and Y.Q. Chen, "Iterative learning approach to a diffusion control problem in an irrigation application", *Proc. IEEE Int. Conf. on Mechatronics and Automation* 1, 1329–1334 (2006).

[20] H. Zhao, *Passive, Iterative, and Repetitive Control for Flexible Distributed Parameter Systems*, Ph.D. Dissertation, The Pennsylvania State University, USA, 2005.