

Application of L^1 -impulse method to the optimization problems in power theory

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Abstract. In optimization power theory we can distinguish the three approaches:

- the theory of instant power values
- the theory of average power values (integral power)
- the theory of instant–average power value.

The theory of instant power uses the instant power and signals values i.e. $p(t) = u(t)i(t)$ whereas the theory of average power uses the energy or average power terms i.e. $P = (u(t), i(t))$ (the dot the product of signals). The main problem in the average power theory comes from the Schwartz inequality:

$$|(u, i)| \leq \|u\| \|i\|,$$

where $\|u\| = \sqrt{(u, u)}$, $\|i\| = \sqrt{(i, i)}$.

This inequality causes numerous optimization problems, among which the norm of the current minimization is the most important one:

$$\begin{aligned} \|i\| &\rightarrow \min, \\ (u, i) - P &= 0. \end{aligned}$$

Whereas the theory of instant-average power values joins both aforementioned methods and uses so-called ‘instant active power’:

$$P(t) = \int_{t-T}^t u(t')i(t')dt'.$$

The mathematic methods used in these theories derive from the theorems of signals and instant power modulation. This article deals only with the average power theory which uses the L^1 impulses as an alternative to the Fourier series method. This technique is efficient when the energy is transmitted with highly distorted periodic signals.

Key words: periodically time-varying networks, operational calculus, stability, synthesis, optimization.

1. Optimization problems in one loop supply system

The optimization problems in power theory refer to the one loop circuit consisting of emf e , linear operator Z of the inner impedance and load impedance operator Z_o . Figure 1 shows that circuit where the power is transmitted by an ideal power line. In the picture the power fluxes i.e. main flux P , source flux (e, i) and loss flux (Ri, i) are marked too [1].

This circuit is analyzed in [2, 3]. The value of average main power flux is the functional of the current i :

$$P(i) = (u, i) = (e, i) - (Ri, i), \quad (1)$$

where $R = \frac{1}{2}(Z + Z^*)$ is the positively definite loss operator of source.

If the signals e, u, i are the elements of the L^1 space then dot product is given by

$$(u, i) = \int_{-\infty}^{\infty} u(t)i(t)dt \quad (2)$$

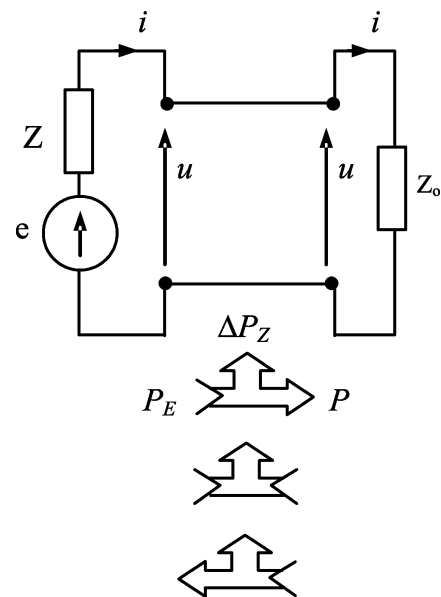


Fig. 1. Supply system and power fluxes with possible flow directions

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and when signals belong to P^T the periodic energy is

$$(u, i) = \int_0^T u(t)i(t)dt \quad (3)$$

or the average energy (average power) is

$$(u, i) = \frac{1}{T} \int_0^T u(t)i(t)dt. \quad (4)$$

The results of above operators are of type $L^1 \rightarrow L^1$ or analogically of type $P^T \rightarrow P^T$ and we assume they belongs to convolution class operators.

The principal optimization task referred to as power matching is the maximization problem of the main power flux:

$$P(i) = (e, i) - (Ri, i) \rightarrow \max, \quad (5)$$

which is solved by operator equation [4, 5]:

$$Ri = \frac{1}{2}e. \quad (6)$$

The existence of solution derive from the fact that R is positively definite. The i_{dop} is termed matching current and is calculated by the inverse operator R^{-1}

$$i_{dop} = \frac{1}{2}R^{-1}e \quad (7)$$

and maximum power value [2]:

$$P_{\max} = \frac{1}{2}(e, i_{dop}) = \frac{1}{4}(r^{-1}e, e). \quad (8)$$

From the Fig. 1 and formula (6) results the ‘impedance’ matching condition

$$Z_o = Z^*. \quad (9)$$

Treating (6) as an operator equation in the L^1 space:

$$R(s)I(s) = \frac{1}{2}E(s),$$

we get Fourier transform of the L^1 -impulse

$$I_{dop}(s) = \frac{1}{2} \frac{E(s)}{R(s)}, \quad (10)$$

where $E(s)$ – Fourier transform of the L^1 -impulse $e(t)$, $R(s) = \frac{1}{2}(Z(s) + Z(-s))$ – source inner loss function for complex value s .

The maximum power value of the source current impulse is given by the dot product

$$\begin{aligned} P_{\max} &= \frac{1}{2} \frac{1}{2\pi j} \int E(s)I_{dop}(-s)ds = \\ &= \frac{1}{4} \frac{1}{2\pi j} \int R^{-1}(s)E(s)E(-s)ds \end{aligned} \quad (11)$$

or

$$P_{\max} = \frac{1}{4} \frac{(-1)}{2\pi j} \int R(s)E(s)E(-s)ds. \quad (12)$$

The time original of the L^1 -impulse $i_{dop}(t)$ we can get using the diagram in fig. (3, 4) [6] and the convolution of type

$h * x$ or the L^1 -isomorphism F^{-1} directly from the $I_{dop}(s)$ transform.

In T -periodic signal domain the result in form of T -periodic extension can be derived in three ways:

1. by means of T -periodic extension of $i_{dop}(t)$ signal: $\tilde{i}_{dop} = \sum_{p=-\infty}^{\infty} i_{dop}(t + pT)$,
2. by means of cyclic convolution,
3. by means of P^T -homomorphism F_T^{-1} directly from the L^1 - impulse transform $I_{dop}(s)$ [6].

From the formula (10) taking into consideration the property of $R(s)$ function it results that the poles of $I_{dop}(s)$ are placed on both sides of imaginary axis and at the same time there are no essential singularities on imaginary axis.

Example 1. The use of the L^1 -impulse method will be shown hear. As an example we take the source with finite duration L^1 -impulse and the inner impedance operator as RC in parallel branch (Fig. 2).

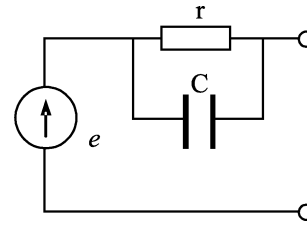


Fig. 2. Equivalent circuit of power source

For the above circuit:

$$Z(s) = \frac{r}{1 + srC},$$

$$R(s) = \frac{r}{1 - (srC)^2},$$

$$R(s)^{-1} = \frac{1}{r}(1 - (srC)^2),$$

$$I_{dop} = \frac{1}{2} \frac{1}{r} (1 - (srC)^2) E(s),$$

thus

$$i_{dop} = \frac{1}{2} \frac{1}{r} \left(e(t) - (rC)^2 \frac{d^2 e}{dt^2} \right)$$

and

$$\begin{aligned} P_{\max} &= \frac{1}{2}(e, i_{dop}) = \frac{1}{4} \frac{1}{r} \left((e, e) - (rC)^2 \left(e, \frac{d^2 e}{dt^2} \right) \right) = \\ &= \frac{1}{4r} \left((e, e) + (rC)^2 \left(\frac{de}{dt}, \frac{de}{dt} \right) \right). \end{aligned} \quad (13)$$

From formula (13) results that the condition of solution existence of

$$P_{\max} = \frac{1}{2}(e, i_{dop}) = \frac{1}{4r} \left(\| e \|^2 + (rC)^2 \left\| \frac{de}{dt} \right\|^2 \right)$$

is the existence of (de/dt) of the source signal. Thus $e(t)$ have to be smooth enough. This is shown in Fig. 3.

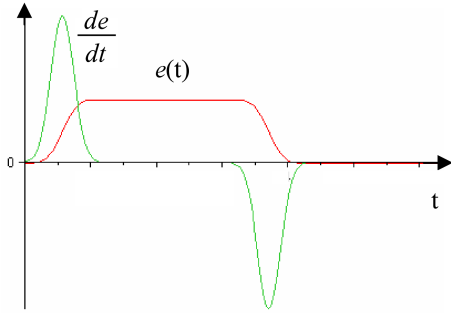


Fig. 3. The waveform of source voltage L^1 -impulse and its derivative versus time

Example 2. In Fig. 4 the power source equivalent circuit is shown with its inner impedance operator. The source voltage square wave has 1 V maximum value and τ duration.

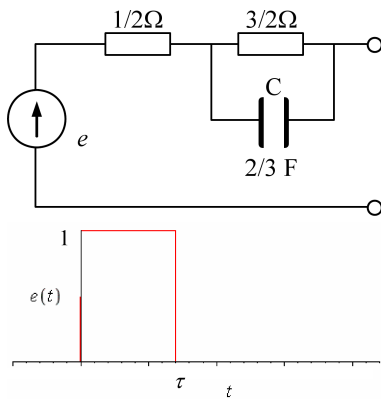


Fig. 4. Equivalent circuit of the power source and its voltage waveform

In this case:

$$Z(s) = \frac{1}{2} \frac{(4 + s)}{1 + s},$$

$$R(s) = \frac{(4 - s^2)}{2(1 - s^2)},$$

$$E(s) = \frac{1}{s}(1 - e^{-s\tau}),$$

$$R(s)^{-1} = 2 \frac{(1 + s)(1 - s)}{(2 + s)(2 - s)}, \quad (14)$$

$$I_{\text{dop}}(s) = \Phi(s)(1 - e^{-s\tau}), \quad (15)$$

where $\Phi(s) = \frac{s^2 - 1}{s(s + 2)(s - 2)}$.

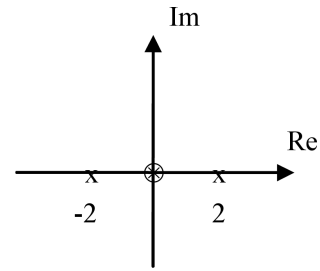


Fig. 5. Poles distribution of $\Phi(s)$

In Fig. 5 the pole distribution of $\Phi(s)$ is shown. Point $s = 0$ is the removable pole of $I_{\text{dop}}(s)$ because:

$$\left[\frac{1}{s}(1 - e^{-s\tau}) \right]_{s \rightarrow 0} \rightarrow \tau.$$

With the Jordan lemma we get:

$$i_{\text{dop}}(t) = \begin{cases} \frac{(-1)}{2\pi j} \oint \Phi(s)(1 - e^{-s\tau})e^{st} ds & t < 0 \\ \frac{1}{2\pi j} \oint \Phi(s)e^{st} ds + \frac{1}{2\pi j} \oint \Phi(s)e^{s(t-\tau)} ds & 0 < t < \tau. \\ \frac{1}{2\pi j} \oint \Phi(s)(1 - e^{-s\tau})e^{st} ds & t > \tau \end{cases}$$

Thus from the Cauchy residue formula results that:

$$i_{\text{dop}} = \begin{cases} -\frac{3}{8}(1 - e^{-2\tau})e^{2t} & t < 0 \\ \frac{1}{4} + \frac{3}{8} [e^{-2t} + e^{2(t-T)}] & 0 < t < \tau \\ -\frac{3}{8}(e^{2\tau} - 1)e^{-2t} & t > \tau \end{cases} \quad (16)$$

and

$$P_{\text{max}} = \frac{1}{2}(e, i_{\text{dop}}) = \frac{1}{2} \int_{-\infty}^{\infty} e(t)i_{\text{dop}}(t)dt = \frac{1}{2} \int_0^T i_{\text{dop}}(t)dt = \tau \left(\frac{1}{8} + \frac{3}{16} \frac{(1 - e^{2\tau})}{\tau} \right).$$

The maximum power value can be also calculated using complex variable integral

$$P_{\text{max}} = \frac{1}{4}(R^{-1}e, e) = \frac{1}{4} \frac{1}{2\pi j} \int_{\uparrow} R^{-1}(s)E(s)E(-s)ds = \frac{1}{2\pi j} \int \frac{s^2 - 1}{(s + 2)(s - 2)} \frac{(e^{s\tau} - 1)}{s^2} ds = \tau \left(\frac{1}{8} + \frac{3}{16} \frac{(1 - e^{-2\tau})}{\tau} \right). \quad (17)$$

The integration curve for (17) is shown in Fig. 6. The integrated function is chosen so as to meet the Jordan lemma condition (21) [6].

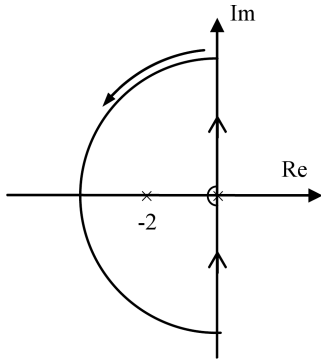


Fig. 6. Integration curve used in (16) integral

Using in turn the isomorphism we get time signal of the L^1 -impulse:

$$R^{-1}(s) = 2 \frac{(1+s)(1-s)}{(2+s)(2-s)} = 2 - \frac{3}{2} \frac{1}{2+s} - \frac{3}{2} \frac{1}{2-s} \leftrightarrow$$

$$\leftrightarrow 2\delta(t) - \frac{3}{2}e^{2t}\mathbf{1}(-t) - \frac{3}{2}e^{-2t}\mathbf{1}(t) = r^{-1}(t). \quad (18)$$

Then the impulse (16) derive from linear convolution

$$i_{\text{dop}}(t) = \frac{1}{2} \int_{-\infty}^{\infty} r^{-1}(t-t')e(t')dt'. \quad (19)$$

In Fig. 7 the T -periodic L^1 -impulse of source voltage with duration $\tau < T$ is shown, achieved using Poisson's formula:

$$\tilde{e}(t) = \sum_{p=-\infty}^{\infty} e(t+pT).$$

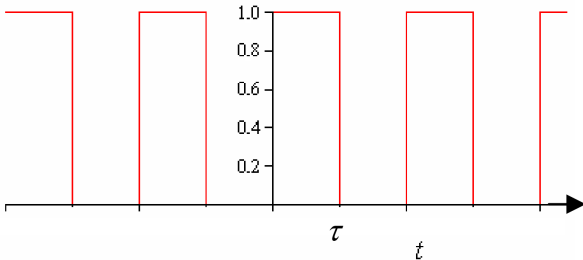


Fig. 7. Periodic L^1 -impulse of source voltage

We can directly calculate the matching current:

$$\begin{aligned} \tilde{i}_{\text{dop}} &= \sum_{p=-\infty}^{\infty} i_{\text{dop}}(t+pT) = \\ &= -\frac{3}{8}(1-e^{-2\tau}) \sum_{p=-\infty}^{\infty} e^{2(t+pT)}\mathbf{1}(-t-pT) + \\ &+ \sum_{p=-\infty}^{\infty} \left(\frac{1}{4} + \frac{3}{8} \left(e^{-2(t+pT)} + e^{2(t=pT-\tau)} \right) \right) \\ & \left(\mathbf{1}(t+pT) - \mathbf{1}(t+pT-\tau) \right) - \frac{3}{8}(e^{2\tau}-1) \\ & \sum_{p=-\infty}^{\infty} e^{-2(t+pT)}\mathbf{1}(t+pT-\tau), \end{aligned}$$

for $0 < t < \tau$

$$\tilde{i}_{\text{dop}} = \frac{1}{4} + \frac{3}{8} \frac{1-e^{-2(T-\tau)}}{1-e^{-2T}} \left(e^{-2t} + e^{2(t-\tau)} \right)$$

and for $\tau < t < T$:

$$\tilde{i}_{\text{dop}}(t) = -\frac{3}{8} \frac{1}{1-e^{-2T}} \left((e^{2\tau}-1)e^{-2t} + (1-e^{-2\tau})e^{2(t-T)} \right), \quad (20)$$

thus

$$P_{\text{max}} = \frac{1}{2}(\tilde{e}, \tilde{i}_{\text{dop}}) = \frac{1}{2} \int_0^T \tilde{e}(t)\tilde{i}_{\text{dop}}(t)dt =$$

$$= \frac{1}{2} \int_0^T \tilde{i}_{\text{dop}}(t)dt = 2 \frac{e^{2\tau}-1}{(1-e^{-2T})^2} e^{-2T}.$$

The calculation of cyclic convolution has the following stages

1. from (18) we get cyclic impulse function

$$\tilde{r}^{-1}(t) = \sum_{p=-\infty}^{\infty} r^{-1}(t+pT)$$

and for $t \in [0, T)$:

$$\begin{aligned} \tilde{r}^{-1}(t) &= 2\delta(t) - \frac{3}{2} \sum_{p=0}^{\infty} e^{-2(t+pT)} - \frac{3}{2} \sum_{p=1}^{\infty} e^{2(t-pT)} = \\ &= 2\delta(t) - \frac{3}{2} \frac{e^{-2t}}{(1-e^{-2T})} - \frac{3}{2} \frac{e^{2(t-T)}}{(1-e^{-2T})} \end{aligned}$$

1. from the formula of cyclic convolution (the equivalent for linear convolution (19)) we get:

$$\begin{aligned} \tilde{i}_{\text{dop}} &= e(t) + \frac{1}{2} \int_0^t \gamma(t-t')e(t')dt' + \\ &+ \frac{1}{2} \int_t^T \gamma(t-t'+T)e(t')dt' = \end{aligned}$$

$$= \begin{cases} 1 + \frac{1}{2} \int_0^t \gamma(t-t')dt' + \frac{1}{2} \int_t^T \gamma(t-t'+T)dt' & 0 < t < \tau \\ \frac{1}{2} \int_0^{\tau} \gamma(t-t')dt' & \text{otherwise} \end{cases}.$$

The integration result is the same as in (20). It is worth to note that:

$$\tilde{i}_{\text{dop}}(t) \rightarrow i_{\text{dop}}(t)$$

for $T \rightarrow \infty$.

Now we must change a bit the formula (47) [6] to use homomorphism F_T^{-1} because the function $I_{\text{dop}}(s)$ (see 15) is quasi-rational i.e. has the time delay part. Such function describes square wave signals or strobe pulses.

When the homomorphism transform (47) has form [6]:

$$Y(s) = H(s) \frac{1-e^{-s\tau}}{s}, \quad (22)$$

then the equivalent L^1 -impulse time form is given by the integral formula of the inverse Fourier transform:

$$y(t) = \left(-\frac{1}{2\pi j} \int H(s) \frac{1 - e^{-s\tau}}{s} e^{st} ds \right) \mathbf{1}(-t) + \left(\frac{1}{2\pi j} \int H(s) \frac{1}{s} e^{st} ds + \frac{1}{2\pi j} \int H(s) \frac{e^{s(t-\tau)}}{s} ds \right) (\mathbf{1}(t) - \mathbf{1}(t - \tau)) + \left(\frac{1}{2\pi j} \int H(s) \frac{1 - e^{-s\tau}}{s} e^{st} ds \right) \mathbf{1}(t - \tau). \quad (23)$$

Especially for a partial fraction

$$Y(s) = \frac{1}{s - \sigma} \frac{1 - e^{-s\tau}}{s}, \quad (24)$$

using (23) we get:

for $\text{Re } \sigma > 0$:

$$y(t) = \left(\frac{-1}{2\pi j} \int \frac{1}{s - \sigma} \frac{1 - e^{-s\tau}}{s} e^{st} ds \right) \mathbf{1}(-t) + \left(\frac{1}{2\pi j} \int \frac{1}{s - \sigma} e^{st} ds + \frac{1}{2\pi j} \int \frac{1}{s - \sigma} \frac{e^{s(t-\tau)}}{s} ds \right) (\mathbf{1}(t) - \mathbf{1}(t - \tau)) = -\frac{1 - e^{-\sigma t}}{\sigma} e^{\sigma t} \mathbf{1}(-t) + \frac{1}{\sigma} (e^{\sigma(t-\tau)} - 1) (\mathbf{1}(t) - \mathbf{1}(t - \tau))$$

and for $\text{Re } \sigma < 0$:

$$y(t) = \left(\frac{1}{2\pi j} \int \frac{1}{s - \sigma} e^{st} ds + \frac{1}{2\pi j} \int \frac{1}{s - \sigma} \frac{e^{s(t-\tau)}}{s} ds \right) [\mathbf{1}(-t) - \mathbf{1}(t - \tau)] + \left(\frac{1}{2\pi j} \int \frac{1}{s - \sigma} \frac{1 - e^{-s\tau}}{s} e^{st} ds \right) \mathbf{1}(t - \tau) = \frac{1 - e^{-\sigma t}}{\sigma} e^{\sigma t} \mathbf{1}(t - \tau) + \frac{1}{\sigma} (e^{\sigma t} - 1) [\mathbf{1}(t) - \mathbf{1}(t - \tau)].$$

For rational function $H(s)$ without poles in infinity:

$$H(s) = H(\infty) - \sum_{\sigma \in \mathcal{P}_+} \frac{a^+(\sigma)}{s - \sigma} + \sum_{\sigma \in \mathcal{P}_-} \frac{a^-(\sigma)}{s - \sigma}$$

we get the L^1 -impulse

$$y(t) = \mathbf{1}(-t) \sum_{\sigma \in \mathcal{P}_+} \frac{a^+(\sigma)}{\sigma} (1 - e^{-\sigma\tau}) e^{\sigma t} = \left(H(\infty) + \sum_{\sigma \in \mathcal{P}_+} \frac{a^+(\sigma)}{\sigma} (1 - e^{\sigma(t-\tau)}) \right) + \sum_{\sigma \in \mathcal{P}_-} \frac{a^-(\sigma)}{\sigma} (e^{\sigma t} - 1) (\mathbf{1}(t) - \mathbf{1}(t - \tau)) + \mathbf{1}(t - \tau) \sum_{\sigma \in \mathcal{P}_-} \frac{a^-(\sigma)}{\sigma} (1 - e^{-\sigma\tau}) e^{\sigma t}. \quad (25)$$

Determining the T -periodic extension of $y(t)$ L^1 -impulse according to the Poisson formula

$$\tilde{y}(t) = \sum_{p=-\infty}^{\infty} y(t + pT),$$

in the range $t \in [0, T)$ we get:

$$\tilde{y}(t) = \sum_{\sigma \in \mathcal{P}_+} \frac{a^+(\sigma)}{\sigma} (1 - e^{-\sigma\tau}) e^{\sigma t} \sum_{p=1}^{\infty} (e^{-\sigma T})^p + H(\infty) + \sum_{\sigma \in \mathcal{P}_+} \frac{a^+(\sigma)}{\sigma} (1 - e^{\sigma(t-\tau)}) + \sum_{\sigma \in \mathcal{P}_-} \frac{a^-(\sigma)}{\sigma} (e^{\sigma t} - 1) + \sum_{\sigma \in \mathcal{P}_-} \frac{a^-(\sigma)}{\sigma} (1 - e^{-\sigma\tau}) e^{\sigma t} \sum_{p=1}^{\infty} (e^{\sigma T})^p = H(\infty) + \sum_{\sigma \in \mathcal{P}_+} \frac{a^+(\sigma)}{\sigma} \left(1 - e^{\sigma(t-\tau)} + \frac{1 - e^{-\sigma\tau}}{1 - e^{-\sigma T}} e^{\sigma(t-T)} \right) + \sum_{\sigma \in \mathcal{P}_-} \frac{a^-(\sigma)}{\sigma} \left(e^{\sigma t} - 1 + \frac{1 - e^{-\sigma\tau}}{1 - e^{-\sigma T}} e^{\sigma(t+T)} \right),$$

when $0 < t < \tau$

and:

$$\tilde{y}(t) = \sum_{\sigma \in \mathcal{P}_+} \frac{a^+(\sigma)}{\sigma} \frac{1 - e^{-\sigma\tau}}{1 - e^{-\sigma T}} e^{\sigma(t-T)} + \sum_{\sigma \in \mathcal{P}_-} \frac{a^-(\sigma)}{\sigma} \frac{1 - e^{-\sigma\tau}}{1 - e^{\sigma T}} e^{\sigma t} \quad (26)$$

when $\tau < t < T$.

It is easy to check that the received formulas (25) and (26) confirm previous results (16) and (20).

Another example of optimization task in power theory is the current norm minimization with assumed main power flux value (see diagram 1.1), [2]:

$$(i, i) \rightarrow \min, \quad (27)$$

$$(e, i) - (Ri, i) = P.$$

It can be proved that the solution of above problem is the family of current signals meeting equation

$$(1 + \lambda R)i = 0.5\lambda e, \quad (28)$$

where λ is the real parameter. For the λ values for which the solution of (28) exists the energy flux function is definite as

$$F(\lambda) = (e, i_\lambda) - (Ri_\lambda, i_\lambda), \quad (29)$$

where

$$i_\lambda = 0.5\lambda(1 + \lambda R)^{-1}e. \quad (30)$$

For the assumed main power flux value P the λ_* values are calculated from

$$F(\lambda) = P \quad (31)$$

and its corresponding current is referred to as optimal current

$$i_{\text{opt}} = 0.5\lambda_*(1 + \lambda_* R)^{-1}e. \quad (32)$$

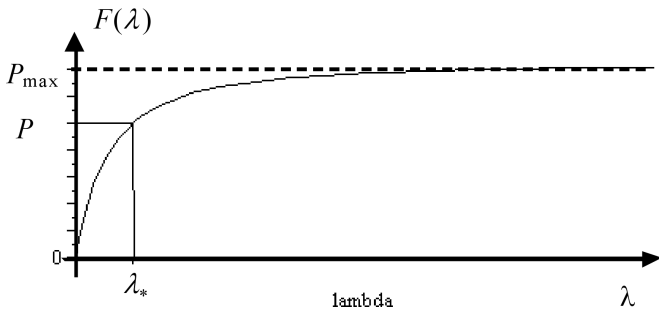


Fig. 8. Main energy flux function versus λ

It can be proved [5] that for $\lambda > 0$ the $F(\lambda)$ function is ascending monotonically and is bounded by P_{\max} value (which is also the solution of maximization task)

$$(e, i) - (Ri, i) \rightarrow \max.$$

Treating (28) analogically, as before, as operator-convolution equation in the L^1 :

$$[1 + \lambda R(s)]I(s) = 0.5\lambda E(s), \quad (33)$$

we get the whole λ -family of the L^1 -impulse current transforms:

$$I_\lambda(s) = \frac{0.5\lambda}{1 + \lambda R(s)} E(s). \quad (34)$$

If $R(s)$ is the rational function i.e.

$$R(s) = \frac{L_R(s)}{M_R(s)}, \quad (35)$$

where $L_R(s), M_R(s)$ are polynomials then

$$I_\lambda(s) = \frac{0.5\lambda M_R(s)}{M_\lambda(s)} E(s) \quad (36)$$

and

$$F(\lambda) = \frac{0.5\lambda}{2\pi j} \int \frac{M_{\lambda|2}(s) M_R(s)}{[M_\lambda(s)]^2} E(s) E(-s) ds, \quad (37)$$

where

$$\begin{aligned} M_\lambda(s) &= M_R(s) + \lambda L_R(s), \\ M_{\lambda|2}(s) &= M_R(s) + \frac{\lambda}{2} L_R(s). \end{aligned} \quad (38)$$

It is easily shown that derivative of $F(\lambda)$ is

$$F'(\lambda) = 0.5 \frac{1}{2} \int \left(\frac{M_R(s)}{M_\lambda(s)} \right)^3 E(s) E(-s) ds \quad (39)$$

and the solution of (31) can be always achieved using Newton's method

$$\delta\lambda = \frac{P - F(\lambda)}{F'(\lambda)} \rightarrow \lambda =: \lambda + \delta\lambda. \quad (40)$$

But this equation can be also resolved using an approximation of $F(\lambda)$ e.g.

$$F(\lambda) \cong P_{\max} \cdot \left(1 - \left(\frac{1}{(1 + a \cdot \lambda)^2} \right) \right), \quad (41)$$

we can calculating a from

$$F'(0) = \left[\frac{2P_{\max}a}{(1 + a\lambda)^3} \right]_{\lambda \rightarrow 0} = 0.5(e, e),$$

then we get the Eq. (31) in form:

$$\frac{1}{(1 + a \cdot \lambda)^2} = 1 - \frac{P}{P_{\max}}$$

and

$$\lambda_* = \frac{1 - \sqrt{1 - \frac{P}{P_{\max}}}}{\sqrt{1 - \frac{P}{P_{\max}}} a} = \frac{1 - \sqrt{1 - \frac{P}{P_{\max}}}}{\sqrt{1 - \frac{P}{P_{\max}}} \frac{0.5(e, e)}{2P_{\max}}}. \quad (42)$$

Example 3. The power source with diagram shown in Fig. 4. (see Example 2) will be now considered. Now we have

$$M_R(s) = 2(1 - s^2),$$

$$M_\lambda(s) = 2 + 4\lambda - (2 + \lambda)s^2,$$

$$M_{\lambda|2} = \left(2 + \frac{\lambda}{2} \right) \left(\frac{4 + 4\lambda}{4 + \lambda} - s^2 \right)$$

and

$$F(\lambda) = a_\lambda \frac{1}{2\pi j} \int \frac{(s^2 - \sigma_\lambda)(s^2 - 1)}{(s - s_\lambda)^2 (s + s_\lambda)^2} ds,$$

where

$$\begin{aligned} A_\lambda &+ \frac{\lambda(4 + \lambda)}{(2 + \lambda)^2}, \\ \sigma_\lambda &= 4 \frac{1 + \lambda}{4 + \lambda}, \end{aligned} \quad (43)$$

$$s_\lambda = -\sqrt{\frac{2 + 4\lambda}{2 + \lambda}}.$$

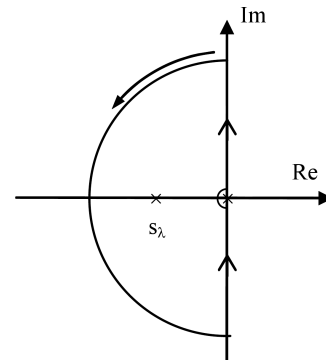


Fig. 9. Contour used to calculate values of $F(\lambda)$

When $\lambda \rightarrow \infty$: $a_\lambda \rightarrow 1, \sigma_\lambda \rightarrow 4, s_\lambda \rightarrow -2$ then (see 17):

$$F(\lambda) \rightarrow \frac{1}{2\pi j} \int \frac{s^2 - 1}{(s + 2)(s - 2)} \frac{e^{s\tau} - 1}{s^2} ds = P_{\max}.$$

The transform of current L^1 -impulse $I_\lambda(s)$ is calculated from (36)

$$I_\lambda = b_\lambda \frac{s^2 - 1}{(s - s_\lambda)(s + s_\lambda)} \frac{1 - e^{-s\tau}}{s}, \quad (44)$$

where $b_\lambda = \frac{\lambda}{2 + \lambda}$.

Using isomorphism (25) we get time function of the L^1 -impulse $i_\lambda(t)$

$$i_\lambda(t) = -c_\lambda(1 - e^{s_\lambda\tau} e^{-s_\lambda t} \mathbf{1}(-t) + \left(\frac{b_\lambda}{s_\lambda^2} + c_\lambda(e^{s_\lambda t} + e^{-s_\lambda(t-\tau)})\right) [\mathbf{1}(t) - \mathbf{1}(t-\tau)] - c_\lambda(e^{-s_\lambda\tau} - 1) e^{s_\lambda t} \mathbf{1}(t-\tau), \quad (45)$$

where

$$c_\lambda = b\lambda \frac{s_\lambda^2 - 1}{2s_\lambda^2} = \frac{3}{4} \frac{\lambda^2}{(2+\lambda)(1+2\lambda)}, \quad (46)$$

$$\frac{b_\lambda}{s_\lambda^2} = \frac{\lambda}{2(1+2\lambda)}.$$

It is easy to check that when $\lambda \rightarrow \infty$: $b_\lambda \rightarrow 1$, $s_\lambda \rightarrow -2$, $c_\lambda \rightarrow 3/8$ then

$$i_\lambda(t) \rightarrow i_{\text{dop}}(t).$$

Received formulas (43), (45), (46) describes migration of $I_\lambda(s)$ roots with λ changes from 0 to ∞ , which is marked in Fig. 10.

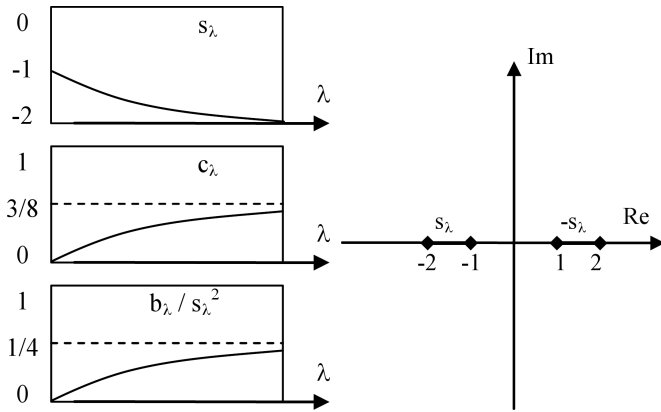


Fig. 10. Root lines in (λ, s_λ) and in Gauss coordinates

For periodic square voltage source (see Fig. 7) the solution of (28) \tilde{i}_λ is given by the Poisson formula

$$\tilde{i}_\lambda(t) = \sum_{p=-\infty}^{\infty} i_\lambda(t + pT).$$

For $t \in [0, T)$:

$$\tilde{i}_\lambda = \frac{b_\lambda}{s_\lambda^2} + c_\lambda \frac{1 - e^{s_\lambda(T-\tau)}}{1 - e^{e_\lambda T}} \left(e^{s_\lambda t} + e^{-s_\lambda(t-\tau)} \right),$$

when $0 < t < \tau$, and

$$\tilde{i}_\lambda = -c_\lambda \frac{1}{1 - e^{e_\lambda T}} \left((e^{-s_\lambda\tau} - 1) e^{s_\lambda t} + (1 - e^{s_\lambda\tau}) e^{-s_\lambda(t-T)} \right),$$

when $\tau < t < T$.

The example of calculus the L^1 -impulse optimal current will be now presented:

$$\text{for } \tau = 1; \quad \frac{P}{P_{\max}} = 0.75$$

the formula (42) is:

$$\lambda_* = \frac{1 - \sqrt{1 - 0.75}}{\sqrt{1 - 0.75}a} = \frac{1 - \sqrt{1 - 0.75}}{0.5} = 1.1485$$

$$= \frac{1 - \sqrt{1 - 0.75}}{2 \left(\frac{1}{8} + \frac{3}{16} (1 - e^{-2}) \right)}$$

thus:

$$s_\lambda = -1.45,$$

$$c_\lambda = 0.314,$$

$$\frac{b_\lambda}{s_\lambda^2} = 0.174,$$

$$i_{\text{opt}} = -0.240 e^{1.45t} \mathbf{1}(-t) + (0.174 + 0.314(e^{-1.45t} + 0.234e^{1.45t})) [\mathbf{1}(t) - \mathbf{1}(t-1)] - 1.02e^{-1.45t} \mathbf{1}(t-1).$$

Another example of power optimization task is the difference voltage minimization defined as a norm of difference between source and terminal voltage when main power flux P is given [2, 3] (see Fig. 1).

$$(e - u, e - u) \rightarrow \min \quad (47)$$

$$(e, i) - (Ri, i) = P.$$

The solution of (47) is a λ -family of signals meeting

$$(Z^*Z + \lambda R)i = 0.5\lambda e. \quad (48)$$

The specimen of current signal which meets the main power flux condition is given by (31).

The special case of (48) is the no-loss source i.e. with zero loss operator:

$$Z^*Zi = 0.5\lambda e. \quad (49)$$

Such equation can be easily solved:

$$i_\lambda = 0.5\lambda Y Y^* e, \quad (50)$$

where Y is the reciprocal of Z – source inner impedance operator.

The Eq. (31) simplifies to

$$0.5\lambda(Y Y^* e, e) = P,$$

thus the optimal current is:

$$i_{\text{opt}} = \frac{P}{Y Y^* e, e} Y Y^* e. \quad (51)$$

Example 4. The quasi lossless source will be now considered. The source having an inner impedance and giving the voltage pulse is shown in Fig. 11.

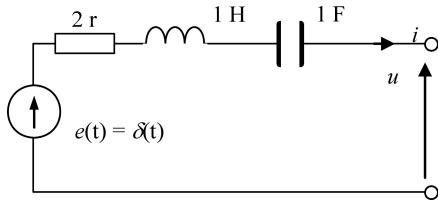


Fig. 11. Diagram of almost lossless source

To move poles away from the imaginary axis the little dissipation caused by the resistance r was assumed.

Then

$$Z(s) = 2r + s + \frac{1}{s} = \frac{s^2 + 2rs + 1}{s}.$$

Assuming $r \ll 1$ we get approximation of zeros of impedance function $Z(s)$:

$$\sigma = -r + j, \quad \sigma^* = -r - j.$$

Thus the YY^* operator is given by

$$Y(s)Y(-s) = \frac{-s^2}{(s - \sigma)(s - \sigma^*)(s + \sigma)(s + \sigma^*)}. \quad (52)$$

The distribution of poles resulting from (52) is shown in Fig. 12.

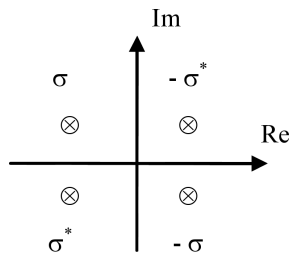


Fig. 12. Poles distribution of source from Fig. 11

The quadratic form (51) has values:

$$\begin{aligned} (YY^* e, e) &= \frac{1}{2\pi j} \int Y(s)Y(-s)E(s)E(-s)ds = \\ &= \frac{1}{2\pi j} \int \frac{-s^2}{(s - \sigma)(s - \sigma^*)(s + \sigma)(s + \sigma^*)} ds = \frac{1}{4r}. \end{aligned}$$

Thus from (51) results transform of the optimal current

$$I_{opt}(s) = 4rP \frac{s^2}{(s - \sigma)(s - \sigma^*)(s + \sigma)(s + \sigma^*)}, \quad (53)$$

now from isomorphism [6] (see 31) or from the curvilinear integral method we get the L^1 -impulse in time domain:

$$\begin{aligned} i_{opt}(t) &= \left(\frac{(-1)}{2\pi j} \int I_{opt}(s)e^{st} ds \right) \mathbf{1}(-t) + \\ &= \left(\frac{1}{2\pi j} \int I_{opt}(s)e^{st} ds \right) \mathbf{1}(t) = \\ &= (Pe^{rt} \cos(t))\mathbf{1}(-t) + (Pe^{-rt} \cos(t))\mathbf{1}(t) = \\ &= \frac{1}{2}P \left(e^{-\sigma t} + e^{-\sigma^* t} \right) \mathbf{1}(-t) + \\ &+ \frac{1}{2}P \left(e^{\sigma t} + e^{\sigma^* t} \right) \mathbf{1}(t). \end{aligned} \quad (54)$$

The equivalent T -periodic extension of the L^1 -impulse (54) can be received by the Poisson formula

$$\begin{aligned} i_{opt}(t) &= \sum_{p=-\infty}^{\infty} i_{opt}(t + pT) = \\ &= \frac{1}{2}P \left(e^{-\sigma t} \sum_{p=-\infty}^{\infty} (e^{\sigma T})^p + e^{-\sigma^* t} \sum_{p=1}^{\infty} (e^{\sigma^* T})^p + \right. \\ &\left. + e^{\sigma t} \sum_{p=0}^{\infty} (e^{\sigma T})^p + e^{\sigma^* t} \sum_{p=0}^{\infty} (e^{\sigma^* T})^p \right) \\ &= \frac{1}{2}P \left(\frac{e^{-\sigma(t-T)}}{1 - e^{\sigma T}} + \frac{e^{-\sigma^*(t-T)}}{1 - e^{\sigma^* T}} + \frac{e^{\sigma t}}{1 - e^{\sigma T}} + \frac{e^{\sigma^* t}}{1 - e^{\sigma^* T}} \right) = \\ &= \frac{P}{|1 - P\sigma T|} \left(e^{r(t-T)} \cos(t - T + \varphi) + e^{-rt} \cos(t + \varphi) \right) \end{aligned}$$

for $0 < t < T$, where $\varphi = \angle(1 - e^{\sigma T})$.

2. Some aspects concerning realization of compensatory and matching circuits

In this chapter we will describe the synthesis problem of the optimal receiver according to an optimal criterion. It means that the receiver should be designed as to assure the optimal working condition of source by minimizing assumed optimum criteria. This can be physically realized by two or four terminal compensatory network connected between source and receiver (see diagrams in Fig. 13).

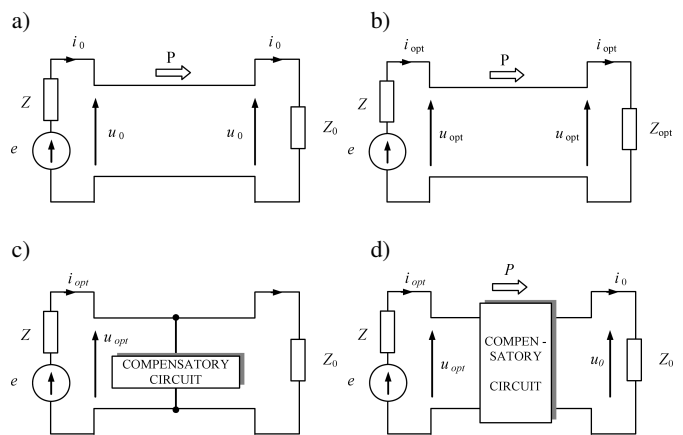


Fig. 13. Compensation concepts: a) circuit to compensate; b) source loaded by an optimal receiver, c) two and d) four-terminal networks matching load to source

The compensation rules will be described below in an example of source which diagram and voltage waveform are shown in Fig. 14.

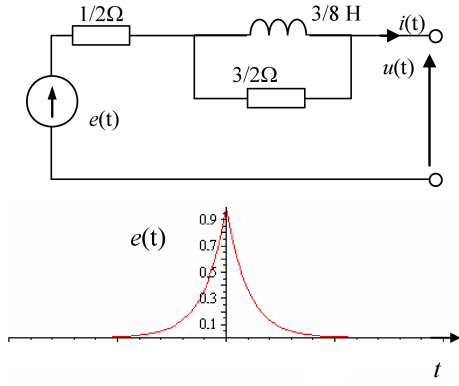


Fig. 14. Example of source and its L^1 -voltage impulse with infinite duration

Now for the synthesis purpose the maximum of the main power flux variant will be chosen.

In this case:

$$Z(s) = 2 \frac{1+s}{4+s},$$

$$R(s) = 2 \frac{(2+s)(2-s)}{(4+s)(4-s)}$$

and for impulse (see Fig. 14)

$$e(t) = e^t \mathbf{1}(-t) + e^{-t} \mathbf{1}(t),$$

the transform is

$$E(s) = \frac{1}{1+s} + \frac{1}{1-s} = \frac{2}{(1+s)(1-s)}.$$

Then the transform of matching current is

$$\begin{aligned} I_{\text{dop}}(s) &= 0.5 R(s)^{-1} E(s) = \\ &= 0.5 \frac{16-s^2}{(1+s)(1-s)(2+s)(2-s)} \end{aligned} \quad (55)$$

and its L^1 -impulse counterpart is

$$\begin{aligned} i_{\text{dop}}(t) &= (1.25e^t - 0.5e^{2t}) \mathbf{1}(-t) + \\ &+ (1.25e^{-t} - 0.5e^{-2t}) \mathbf{1}(t). \end{aligned} \quad (56)$$

From Poisson formula we get T -periodic extension of signals

$$\tilde{e}(t) = \sum_{p=-\infty}^{\infty} e(t+pT) = \sum_{p=1}^{\infty} e^{t-pT} + \quad (57)$$

$$+ \sum_{p=0}^{\infty} e^{-(t+pT)} = \frac{1}{1-e^{-T}} (e^{t-T} + e^{-t}),$$

$$\tilde{i}_{\text{dop}}(t) = \sum_{p=-\infty}^{\infty} i_{\text{dop}}(t+pT) = \sum_{p=1}^{\infty} (1.25e^{(t-pT)} - 0.5e^{2(t-pT)}) +$$

$$+ \sum_{p=0}^{\infty} (1.25e^{-(t+pT)} - 0.5e^{-2(t+pT)}) =$$

$$= \frac{1.25}{1-e^{-T}} (e^{t-T} + e^{-t}) - \frac{0.5}{1-e^{-2T}} (e^{2(t-T)} + e^{-2t}), \quad (58)$$

for $t \in [0, T]$.

Thus the maximum energy value is

$$P_{\text{max}} = 0.5 \int_{-\infty}^{\infty} e(t) i_{\text{dop}}(t) dt = \frac{11}{24}. \quad (59)$$

$$\begin{aligned} \tilde{P}_{\text{max}} &= 0.5 \int_0^T \tilde{e}(t) \tilde{i}_{\text{dop}}(t) dt = \\ &= \frac{5}{8} \frac{1}{(1-e^{-T})^2} (2Te^{-T} + 1 - e^{-2T}) - \end{aligned}$$

$$- \frac{1}{4} \frac{1}{(1-e^{-T})(1-e^{2T})} \left(\frac{2}{3} (1-e^{-3T}) + 2e^{-T}(1-e^{-T}) \right). \quad (60)$$

When $T \rightarrow \infty$, $\tilde{P}_{\text{max}} \rightarrow P_{\text{max}}$.

The all other signals can be calculated using the convolution (see diagrams 2.1 (a) and (c))

$$Z i_{\text{dop}}(t) = \int_{-\infty}^{\infty} z(t-\tau) i_{\text{dop}}(\tau) d\tau,$$

where

$$Z(s) = 2 \frac{1+s}{4+s} = 2 - \frac{6}{4+s} \rightarrow z(t) = 2\delta(t) - 6e^{-4t} \mathbf{1}(t).$$

Thus the L^1 -impulse:

$$Z i_{\text{dop}}(t) = (e^t - 0.5e^{2t}) \mathbf{1}(-t) + 0.5e^{-2t} \mathbf{1}(t) \quad (61)$$

and its T -periodic extension:

$$\begin{aligned} \tilde{Z} i_{\text{dop}}(t) &= \sum_{p=-\infty}^{\infty} Z i_{\text{dop}}(t+pT) = \\ &= \frac{1}{1-e^{-T}} e^{t-T} - \frac{0.5}{1-e^{-2T}} e^{2(t-T)} + \frac{0.5}{1-e^{-2T}} e^{-2t} \end{aligned} \quad (62)$$

and

$$\begin{aligned} \tilde{u}_{\text{dop}}(t) &= \tilde{e}(t) - \tilde{Z} i_{\text{dop}}(t) = \\ &= \frac{0.5}{1-e^{-2T}} (e^{2(t-T)} - e^{-2t}) + \frac{1}{1-e^{-T}} e^{-t} \end{aligned} \quad (63)$$

for $t \in [0, T]$.

The synthesis process is shown in Fig. 15.

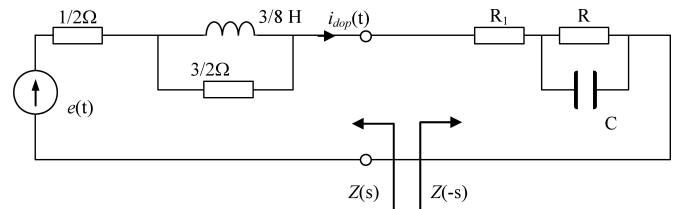


Fig. 15. Synthesis of matching load in $\pm R, C$ class realizing adjoint operator

The matching problem with respect to the maximum main energy flux is met by the load described by the adjoint operator:

$$\frac{R + R_1 + RR_1Cs}{1 + RCs} = Z(-s) = \frac{\frac{1}{2} - \frac{1}{2}s}{1 - \frac{1}{4}s}, \quad G = \frac{1}{R}.$$

Thus comparing coefficients of the rational functions we get set of equations for parameters:

$$\begin{aligned} R_1 + R &= \frac{1}{2}, \\ RR_1C &= -\frac{1}{2}, \\ RC &= -\frac{1}{4}. \end{aligned}$$

It has unequivocal solution:

$$\begin{aligned} R &= -\frac{3}{2}, \\ R_1 &= 2, \\ C &= \frac{1}{6}. \end{aligned}$$

The diagram of source with matching load is shown in Fig. 16.

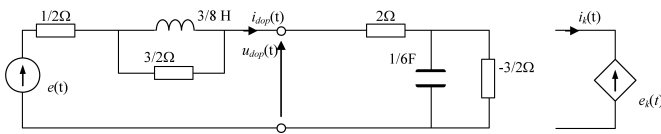


Fig. 16. Matching circuit with negative resistance

The drawback of that matching circuit is a negative resistance. It can be realized by a system with positive feedback e.g. by separately excited DC generator (see Fig. 17a). The armature of motor as a controlled voltage source $e_k(t)$ is put in place of negative resistance.

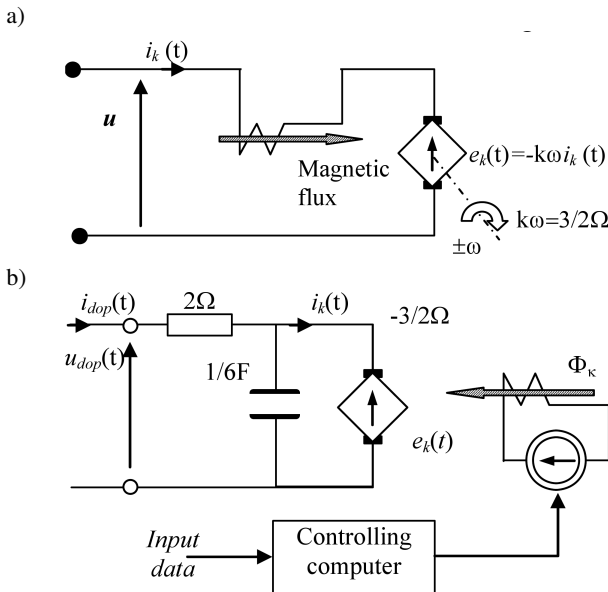


Fig. 17. a) Separately excited DC generator as the negative resistance realization and b) matched load realized with 'intelligent' voltage controlled source

However, such realization of matching circuit (Figs. 16 and 17a) is impractical because is not stable. The main source current in a transient state [5]

$$i(t) = \mathbf{1}(t) \int_0^T \tilde{y}(t \ominus t') \tilde{e}(t') dt' + i_{tr}(t) = \tilde{i}_{dop}(t) + i_{tr}(t)$$

has infinitely ascending transient component:

$$i_{tr}(t) = \sum_{\sigma \in \mathcal{P}(Y)} a(\sigma) e^{\sigma t} \mathbf{1}(t), \quad (64)$$

where:

$$Y(s) = [R(s)]^{-1} = \frac{1}{2} \frac{s^2 - 16}{(s - 2)(s + 2)} - \text{the admittance seen}$$

from the output terminals of source,

$\mathcal{P}(Y)$ – set of $Y(s)$ poles,

$$a(\sigma) = \left[\frac{Y(s)}{1 - e^{-sT}} E(s)(s - \sigma) \right]_{s \rightarrow \sigma} - \text{residue,}$$

$$E(s) = \int_0^T \tilde{e}(t) e^{-st} dt - \text{the finite Laplace transform.}$$

Supply source is then instable circuit which ascending transient component of current (64) is

$$i_{tr}(t) = (a(2)e^{2t} + a(-2)e^{-2t}) \mathbf{1}(t),$$

where

$$\begin{aligned} a(2) &= \frac{\left(-\frac{3}{2}\right)}{1 - e^{-2T}} \int_0^T \tilde{e}(t) e^{-2t} dt = \\ &= \frac{\left(-\frac{3}{2}\right)}{(1 - e^{-T})(1 - e^{-2T})} \left(\frac{1}{3} + e^{-T} - e^{-2T} - \frac{1}{3}e^{-3T} \right), \end{aligned}$$

$$\begin{aligned} a(-2) &= \frac{\left(-\frac{3}{2}\right)}{e^{2T} - 1} \int_0^T \tilde{e}(t) e^{2t} dt = \\ &= \frac{\left(-\frac{3}{2}\right)}{(1 - e^{-T})(e^{2T} - 1)} \left(\frac{1}{3}e^{2T} + e^T - \frac{1}{3}e^{-T} - 1 \right). \end{aligned}$$

The stable, correct working condition of source can be ensured by 'intelligent' controlled source which provide the T -periodic voltage signal (see Fig. 17b):

$$\begin{aligned} \tilde{e}_k(t) &= \tilde{u}_{dop}(t) - 2i_{dop}(t) = \\ &= \frac{1}{1 - e^{-2T}} (1.5e^{2(t-T)} + 0.5e^{-2t}) - \\ &\quad - \frac{1}{1 - e^{-T}} (2.5e^{(t-T)} + 1.5e^{-t}) \end{aligned}$$

for $t \in [0, T)$ (see (57), (58), (62), (63)).

In this case couple of emf connected series-opposing 'see' the stable circuit as shown in Fig. 18.

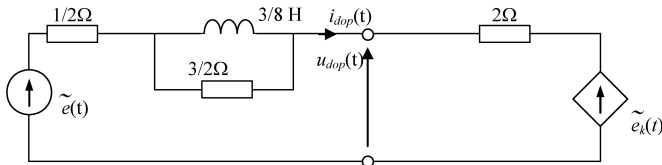


Fig. 18. The circuit with ‘intelligent’ matching source to examine the stability

The integral impedance operator is given by:

$$Z_1(s) = \frac{2 + 2s}{4 + s} + 2 = \frac{s + 2.5}{s + 4}$$

and transient component is

$$I_{tr}(t) = a(-2.5)e^{-2.5t}\mathbf{1}(t),$$

where

$$\begin{aligned} a(-2.5) &= \frac{1}{4} \left(\frac{s + 4}{1 - e^{-sT}} (E(s) - E_k(s)) \right)_{s \rightarrow -2.5} = \\ &= \frac{0.375}{e^{2.5T} - 1} (E_k - E)(-2.5), \end{aligned}$$

$$\begin{aligned} (E_k - E)(-2.5) &= \int_0^T (\tilde{e}_k(t) - \tilde{e}(t)) e^{2.5t} dt = \\ &= \frac{1}{1 - e^{-2T}} (0.3e^{2.5T} + 0.3e^{-2T} + e^{0.5} - 1) - \\ &\quad - \frac{1}{1 - e^{-T}} (e^{2.5T} + 1.7e^{1.5T} - e^{-T} - 1.7). \end{aligned}$$

Of course the modern controlled sources doesn't use electro-motors components in action but power-electronic ones. The model of such circuits derived from [3] and [7] is shown in Fig. 19.

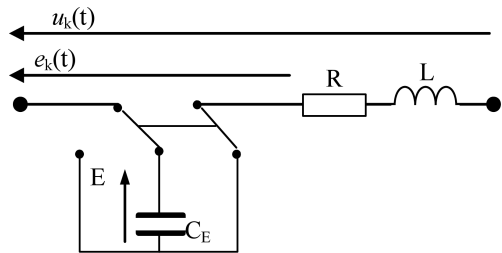


Fig. 19. Realization of the so-called universal branch

The so-called universal compensation branch consists of energy storage component and a two polar switch. By the pulse width modulation we can achieve any signal $e_k(t)$. The RL part smoothes the current.

3. Conclusions

In the article the two principal optimization problems of the so-called average power theory was described. One of these is the problem of the maximization of the source power delivered to the receiver and the second is the minimization of the current norm while the assumed power is delivered to the load. This problem was solved by the new method of L^1 -impulses [6], instead of still commonly used Fourier series method. The advantage of the L^1 -impulse method over the Fourier series method is noticeable when the energy is transmitted with highly distorted periodic signals.

In the article also the new approach to the synthesis of a compensatory circuit which assures the optimal current-voltage signals of the source was presented.

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