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Stability of positive continuous-time linear systems with delays

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Abstract. The asymptotic stability of positive continuous-time linear systems with delays is addressed. It is shown that: 1) the asymptotic stability of the positive systems with delays is independent of their delays, 2) the checking of the asymptotic stability of the positive systems with delays can be reduced to checking of the asymptotic stability of positive systems without delays. Simple stability conditions for the positive systems with delays are give and illustrated by numerical examples.

Key words: positive, continuous-time, system with delays, stability.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial states remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in monographs [1, 2]. The stability and the robust stability of positive discrete-time linear systems without delays and with delays have been investigated in [1-19].

In this paper the stability of positive continuous-time linear systems with delays will be considered. It will be shown that the asymptotic stability of positive continuous-time linear systems is independent of their delays and checking of asymptotic stability of the system with delays can be reduced to checking of the stability of positive systems without delays.

The paper is organized as follows. In Sec. 2 necessary and sufficient conditions for the positivity of the continuous-time systems with delays are established. The main result of the paper is given in Sec. 3, where it is shown that the asymptotic stability of the positive systems with delays is independent of their delays It is also shown that the checking of stability of the systems with delays can be reduced to checking of the stability of positive systems without delays. Concluding remarks are given in Sec. 4.

In this paper the following notation will be used. The set of $n \times m$ real matrices with nonnegative entries will be denoted by $R_+^{n\times m}$ and the set of $n \times n$ Metzler matrices by M_n. A real matrix $A = [a_{ij}] \in R^{n\times n}$ is called Metzler matrix if $a_{ij} \geq 0$ for $i \neq j$. A matrix $A = [a_{ij}] \in R^{n\times n}$ (vector x) is called strictly positive and denoted by $A > 0$ ($x > 0$) if and only if all its entries are positive, i.e. $a_{ij} > 0$ for $i = 1, \dots, n$; $j = 1, \dots, m$. The identity $n\times n$ matrix will be denoted by I_n.

2. Positive continuous-time systems with delays

Consider the continuous-time linear system with q delays in state

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t - d_k) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^p$ are the state, input and output vectors $A_k, k = 0, 1, ..., q$; B, C, D are real matrices of appropriate dimensions and d_k $k = 1, 2, ..., q$ are delays ($d_k \geq 0$). The initial conditions for (1a) has the form

$$x(t) = x_0(t) \quad \text{for} \quad t \in [-d, 0], \quad d = \max_k d_k \tag{2}$$

where $x_0(t)$ is a given vector function.

Definition 1. The system (1) is called (internally) positive if and only if $x(t) \in R_+^n$, $y(t) \in R_+^p$ for any $x_0(t) \in R_+^n$ and for all inputs $u(t) \in R_+^m$, $t \geq 0$.

Theorem 1. The system (1) is (internally) positive if and only if

$$\begin{array}{lll} A_0 \in M_n, & A_k \in R_+^{n\times n}, & k = 1, ..., q, \\ B \in R_+^{n\times m}, & C \in R_+^{p\times n}, & D \in R_+^{p\times m}. \end{array} \tag{3}$$

Proof. Necessity. The equation (1) for $x_0(t) = 0$, $t \in [-d, 0]$ and $u(t) = 0$, $t \geq 0$ takes the form

$$\dot{x}(t) = A_0 x(t), \qquad t \in [0, d]. \tag{4}$$

It is well-known [10, 13] that $x(t) \in R_+^n$ of (4) only if $A_0 \in M_n$. Assuming (1a) $u(t) = 0$, $t \geq 0$, $x_0(d_k) = e_i$, $i = 1, ..., n$ (ith column of I_n) $x(d_j) = 0$, $j = 1, ..., k-1, k+1, ..., n$ for $t = 0$ we obtain $\dot{x}(0) = A_k e_i = A_{ki} \in R_+^n$ where A_{ki} is the ith column of A_k. Hence $A_k \in R_+^{n\times n}$, $k = 1, ..., q$. From (1a) for $t = 0$ and $x_0(t) = 0$, $t \in [-d, 0]$ we have $\dot{x}(0) = Bu(0)$ and $B \in R_+^{n\times m}$ since by definition $u(0) \in R_+^m$ is arbitrary. The necessity of $C \in R_+^{m\times n}$ and $D \in R_+^{p\times m}$ can be shown in the same way as for positive systems without delays [10, 13].

Sufficiency. The solution of (1a) for $t \in [0, d]$ has the form

$$x(t) = e^{A_0 t} + \int_0^t e^{A_0(t-\tau)} \left(\sum_{k=1}^{q} A_k x_0(\tau - d_k) + Bu(\tau) \right) d\tau. \tag{5}$$

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Taking into account that $e^{A_0 t} \in R_+^{n\times n}$, $t \geq 0$ if $A_0 \in M_n$ and the conditions (3), from (5) we obtain $x(t) \in R_+^n$, $t \in [0,d]$ since $x_0(t) \in R_+^n$, $t \in [-d,0]$ and $u(t) \in R_+^m$, $t \geq 0$. From (1b) we have $y(t) \in R_+^p$, $t \in [0,d]$ since $x(t) \in R_+^n$ and $u(t) \in R_+^m$. Using the step method we can extend the considerations for the intervals $[d, 2d]$, $[2d, 3d]$, ...

The positive system (1) is called asymptotically stable if and only if the solution of (1a) for $u(t) = 0$ satisfies the condition $\lim\limits_{t\to\infty} x(t) = 0$ for $x_0(t) \in R_+^n$, $t \in [-d,0]$.

Definition 2. Let to the positive asymptotically stable system (1) a constant input be applied, $u(t) = u \in R_+^m$. A vector $x_e \in R_+^n$ satisfying the equality

$$0 = \sum_{k=0}^{q} A_k x_e + Bu \tag{6}$$

is called the equilibrium point of the system (1) corresponding to the input u.

If the positive system (1) is asymptotically stable then the matrix

$$A = \sum_{k=0}^{q} A_k \in M_n \tag{7}$$

is nonsingular and from (6) we have

$$x_e = -A^{-1}Bu. \tag{8}$$

Theorem 2. The equilibrium point x_e corresponding to strictly positive $Bu > 0$ of the positive asymptotically stable system (1) is strictly positive, i.e. $x_e > 0$.

Proof. Proof is accomplished by contradiction. Suppose that $x_e = 0$ then from (6) we have $Bu = 0$. This contradicts the assumption $Bu > 0$.

Remark 1. For the positive asymptotically stable system (1)

$$-A^{-1} \in R_+^{n\times n}. \tag{9}$$

This follows immediately from (8) since $x_0 \in R_+^n$ and $Bu \in R_+^n$ is arbitrary.

3. Asymptotic stability

Theorem 3. The positive system (1) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in R_+^n$ satisfying the equality

$$A\lambda < 0, \qquad A = \sum_{k=0}^{q} A_k. \tag{10}$$

Proof. First we shall shown that if the system (1) is asymptotically stable then there exists a strictly positive vector $\lambda \in R_+^n$ satisfying (10). Integrating the equation (1a) for $B = 0$ in the interval $[0,\infty]$ we obtain

$$\int_0^\infty \dot{x}(t)dt = A_0 \int_0^\infty x(t)dt + \sum_{k=1}^{q} A_k \int_0^\infty x(t-d_k)dt$$

and

$$x(\infty) - x(0) - \sum_{k=1}^{q} A_k \int_{-d_k}^{0} x(t)dt = A \int_0^\infty x(t)dt. \tag{11}$$

For asymptotically stable positive system

$$x(\infty) = 0,$$

$$x(0) + \sum_{k=1}^{q} A_k \int_{-d_k}^{0} x(t)dt > 0,$$

$$\int_0^\infty x(t)dt > 0$$

and from (11) we have (10) for $\lambda = \int_0^\infty x(t)dt$.

Now we shall show that if (10) holds then the positive system (1) is asymptotically stable. It is well-known that the system (1) is asymptotically stable if and only if the corresponding transpose system

$$\dot{x}(t) = A_0^T x(t) + \sum_{k=1}^{q} A_k^T x(t-d_k) \tag{12}$$

is asymptotically stable. As a candidate for a Lyapunov function for the positive system (12) we chose the function

$$V(x) = x^T\lambda + \sum_{k=1}^{q} \int_{t-d_k}^{t} x^T(\tau)d\tau A_k\lambda \tag{13}$$

which is positive for any nonzero $x(t) \in R_+^n$. Using (13) and (12) we obtain

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T(t)\lambda + \sum_{k=1}^{q} \big(x^T(t) - x^T(t-d_k)\big)A_k\lambda = \\ &= x^T(t)A_0\lambda + \sum_{k=1}^{q} x^T(t-d_k)A_k\lambda \\ &+ \sum_{k=1}^{q} \big(x^T(t) - x^T(t-d_k)\big)A_k\lambda = x^T(t)A\lambda.\end{aligned} \tag{14}$$

If the condition (10) holds then from (14) we have $\dot{V}(x) < 0$ and the system (1) is asymptotically stable.

Remark 2. As a strictly positive vector λ the equilibrium point (8) of the system can be chosen, since

$$A\lambda = A\left(-A^{-1}Bu\right) = -Bu < 0 \quad \text{for} \quad Bu > 0. \tag{15}$$

Theorem 4. The positive system with delays (1) is asymptotically stable if and only if the positive system without delays

$$\dot{x} = Ax, \quad A = \sum_{k=0}^{q} A_k \in M_n \tag{16}$$

is asymptotically stable.

Proof. In [15] it was shown that the positive system (16) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in R_+^n$ such that (10) holds. Hence by Theorem 3 the positive system (1) is asymptotically stable if and only if the positive system (16) is asymptotically stable.

From Theorem 4 it follows that the checking of the asymptotic stability of positive systems with delays (1) can be reduced to checking the asymptotic stability of corresponding positive systems without delays (16). To check the asymptotic stability of positive system (1) the following theorem can be used [2, 16].

Theorem 5. The positive system with delays (1) is asymptotically stable if and only if one of the following equivalent conditions holds:

1. Eigenvalues $s_1, s_2, ..., s_n$ of the matrix A have negative real parts, $\text{Re}s_k < 0$, $k = 1, \ldots, n$.
2. All coefficients of the characteristic polynomial of the matrix A are positive.
3. All leading principal minors of the matrix

$$-A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ldots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{bmatrix} \tag{17}$$

are positive, i.e.

$$|a_{11}| > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, ..., \det[-A] > 0 \tag{18}$$

Example 1. Using the conditions 2) and 3) of Theorem 5 check the asymptotic stability of the positive system (1) for $q = 1$ with the matrices

$$A_0 = \begin{bmatrix} -1 & 0.3 \\ 0.2 & -1.4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}. \tag{19}$$

The characteristic polynomial of the matrix

$$A = A_0 + A_1 = \begin{bmatrix} -0.5 & 0.4 \\ 0.4 & -0.6 \end{bmatrix},$$

has the form

$$\det[I_n s - A] = \begin{vmatrix} s+0.5 & -0.4 \\ -0.4 & s+0.6 \end{vmatrix} = s^2 + 1.1s + 0.14 \tag{20}$$

and its coefficients are positive.
Leading principal minors of the matrix

$$-A = \begin{bmatrix} 0.5 & -0.4 \\ -0.4 & 0.6 \end{bmatrix}$$

are positive, since $\Delta_1 = 0.5$, $\det[-A] = 0.14$.
Therefore, the conditions 2) and 3) of Theorem 5 are satisfied and the positive system (1) with (19) is asymptotically stable.

Theorem 6. The positive system with delays (1) is unstable for any matrices A_k, $k = 1, ..., q$ if the positive system

$$\dot{x} = A_0 x \tag{21}$$

is unstable.

Proof. It is based on Theorem 3. If the system (21) is unstable then does not exist a strictly positive vector $\lambda \in R_+^n$ such that $A_0\lambda < 0$. In this case a strictly positive vector $\lambda \in R_+^n$ satisfying the inequality (10) does not exist, since for the positive system $A_k \in R_+^{n\times n}$ and $A_k\lambda \in R_+^n$, $k = 1, ..., q$.

Theorem 7. If at least one diagonal entry of the matrix A_0 is positive then the positive system (1) is unstable for any A_k, $k = 1, ..., q$.

Proof. It is well-known [10, 16] that the positive system (21) is unstable if at least one diagonal entry of the matrix A_0 is positive. By Theorem 6 the positive system (1) is unstable if the positive system (21) is unstable for any matrices A_k, $k = 1, ..., q$.

Example 2. We shall show that the positive system (1) with

$$A_0 = \begin{bmatrix} -1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \quad \text{and any matrices} \quad A_k \in R_+^{2\times 2}, \ k = 1, ..., q \tag{22}$$

is unstable.

The characteristic polynomial of the matrix (22) of the form

$$\det[I_n s - A_0] = \begin{vmatrix} s+1 & -0.4 \\ -0.3 & s-0.2 \end{vmatrix} = s^2 + 0.8s - 0.32$$

has one negative coefficient. By the condition 2) of Theorem 5 the system (21) with (22) is unstable. Therefore, by Theorem 6 the system (1) with (22) is unstable for any $A_k \in R_+^{2\times 2}$, $k = 1, ..., q$. The same result follows immediately from Theorem 7 since the matrix (22) has one positive diagonal entry ($a_{22} = 0.2$).

Remark 3. It is impossible to stabilize the positive unstable system (1) by changing the matrices $A_k \in R_+^{n\times n}$, $k = 1, 2, ..., q$.

4. Concluding remarks

The asymptotic stability of positive continuous-time linear systems with q delays in state has been addressed. Necessary and sufficient conditions for the positivity of the systems with delays have been established (Theorem 1). It has been shown that the asymptotic stability of the positive continuous-time linear systems with delays is independent of their delays (Theorem 3, 4). The checking of the asymptotic stability of the positive systems with delays can be reduced to checking of the asymptotic stability of the positive systems without delays. Simple equivalent conditions for checking the asymptotic stability of the positive systems have been given in Theorem 5. It has been also shown (Theorem 6) that the positive systems with delays (1) is unstable if the positive systems (21) is unstable or if at least one diagonal entry of the matrix A_0 is positive (Theorem 7). It has been shown that it is impossible to stabilize the positive unstable system with delays (1) by changing of the matrices A_k, $k = 1, 2, ..., q$. The considerations have been illustrated by two numerical examples.

Considerations can be extended for 2D positive hybrid linear systems with delays. An extension of those considerations for positive 2D continuous-time linear systems is an open problem.

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