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on the occasion of his eightieth birthday

Asymptotic stability of positive 2D linear systems with delays

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Abstract. New necessary and sufficient conditions for the asymptotic stability of positive 2D linear systems with delays described by the general model, Fornasini-Marchesini models and Roesser model are established. It is shown that checking of the asymptotic stability of positive 2D linear systems with delays can be reduced to the checking of the asymptotic stability of corresponding positive 1D linear systems without delays. The efficiency of the new criterions is demonstrated on numerical examples.

Key words: 2D systems with delays, asymptotic stability, positive systems, criterion.

1. Introduction

The most popular models of two-dimensional (2D) linear systems are the models introduced by Roesser [1], Fornasini-Marchesini [2, 3] and Kurek [4]. The models have been extended for positive systems in [5–8]. An overview of 2D linear systems theory is given in [9–13], and some recent result in positive systems has been given in monographs [8, 14]. Reachability and minimum energy control of positive 2D systems with one delay in states have been considered in [7]. The choice of the Lyapunov functions for positive 2D Roesser model has been investigated in [15].

The notion of internally positive 2D system (model) with delays in states and in inputs has been introduced and necessary and sufficient conditions for the internal positivity, reachability, controllability, observability and the minimum energy control problem have been established in [16].

The realization problem for 1D positive discrete-time systems with delays has been analyzed in [17, 18] and for 2D positive systems in [19].

The internal stability and asymptotic behavior of 2D positive systems have been investigated by Valcher in [6].

The stability of 2D positive systems described by the Roesser model and synthesis of state-feedback controllers have been considered in the paper [20]. The asymptotic stability of positive 2D linear systems has been investigated in [21].

In this paper new necessary and sufficient conditions for the asymptotic stability of positive 2D linear systems with delays described by the general model, Fornasini-Marchesini models and Roesser model will be established. It will be shown that the checking of the asymptotic stability of positive 2D linear systems with delays can be reduced to the checking of the asymptotic stability of corresponding positive 1D linear systems without delays.

The paper is organized as follows. In Sec. 2 basic definitions and theorems concerning the positive 1D and 2D linear systems are given. In Sec. 3 basic theorems concerning the asymptotic stability of 2D linear systems are presented. New necessary and sufficient conditions for the asymptotic stability of the positive 2D systems with delays are established in Sec. 4. Concluding remarks are given in Sec. 5.

The following notation will be used. $R^{n\times m}$ denotes the set of $n \times m$ real matrices. The set of real $n \times m$ matrices with nonnegative entries will be denoted by $R_+^{n\times m}$ and the set of nonnegative integers will be denoted by Z_+. The $n \times n$ identity matrix will be denoted by I_n.

2. Preliminaries

2.1. Positive 1D systems. Consider the linear discrete-time system:

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i \qquad i \in Z_+ \\ y_i &= Cx_i + Du_i, \end{aligned} \tag{1}$$

where $x_i \in R^n$, $u_i \in R^m$, $y_i \in R^p$ are the state, input and output vectors and, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$.

The system (1) is called (internally) positive if $x_i \in R_+^n$, $y_i \in R_+^p$, $i \in Z_+$ for any $x_0 \in R_+^n$ and every $u_i \in R_+^m$, $i \in Z_+$.

Theorem 1 [8, 14]. The system (1) is positive if and only if

$$A \in R_+^{n\times n}, \quad B \in R_+^{n\times m}, \quad C \in R_+^{p\times n}, \quad D \in R_+^{p\times m}. \tag{2}$$

The positive system (1) is called asymptotically stable if the solution

$$x_i = A^i x_0, \tag{3}$$

of the equation

$$x_{i+1} = Ax_i, \quad A \in R_+^{n\times n}, \quad i \in Z_+, \tag{4}$$

satisfies the condition

$$\lim_{i\to\infty} x_i = 0 \quad \text{for every} \quad x_0 \in R_+^n. \tag{5}$$

Theorem 2 [8, 14]. For the positive system (4) the following statements are equivalent:

1. The system is asymptotically stable,
2. Eigenvalues $z_1, z_2, \ldots, z_n$ of the matrix A have moduli less than 1, i.e. $|z_k| < 1$ for $k = 1, \ldots, n$,

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3. $\det[I_n - zA] \neq 0$ for $|z| \leq 1$,
4. $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of the matrix A defined as $\rho(A) = \max_{1\leq k\leq n} \{|z_k|\}$.

Theorem 3 [8]. The positive system (4) is asymptotically stable if and only if one of the following conditions is satisfied:

1. All coefficients of the characteristic polynomial of the matrix $\hat{A} = A - I_n$

$$w_{\hat{A}}(z) = \det\left[I_n z - \hat{A}\right] = z^n + \hat{a}_{n-1}z^{n-1} + \cdots + \hat{a}_1 z + \hat{a}_0 \tag{6}$$

are positive.

2. All principal minors of the matrix

$$\overline{A} = I_n - A = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \overline{a}_{21} & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{n1} & \overline{a}_{n2} & \cdots & \overline{a}_{nn} \end{bmatrix} \tag{7}$$

are positive, i.e.

$$|\overline{a}_{11}| > 0, \begin{vmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{21} & \overline{a}_{22} \end{vmatrix} > 0, \ldots, \det \overline{A} > 0. \tag{8}$$

Theorem 4 [8]. The positive system (4) is unstable if at least on diagonal entry of the matrix A is greater than 1.

2.2. Positive 2D systems. Consider the general model of 2D linear systems

$$\begin{aligned} x_{i+1,j+1} = A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{i,j} \\ + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \end{aligned} \tag{9a}$$

$$y_{i,j} = Cx_{i,j} + Du_{i,j} \qquad i,j \in Z_+, \tag{9b}$$

where $x_{i,j} \in R^n$, $u_{i,j} \in R^m$, $y_{i,j} \in R^p$ are the state, input and output vectors at the point (i,j) and $A_k \in R^{n\times n}$, $B_k \in R^{n\times m}$,

$$k = 0,1,2, \quad C \in R^{p\times n}, \quad D \in R^{p\times m}.$$

Boundary conditions for (9a) have the form

$$x_{i,0} \in R^n, \quad i \in Z_+ \quad \text{and} \quad x_{0,j} \in R^n, \quad j \in Z_+. \tag{10}$$

The model (9) is called (internally) positive if $x_{i,j} \in R_+^n$ and $y_{i,j} \in R_+^p$, $i,j \in Z_+$ for all boundary conditions $x_{i,0} \in R_+^n$, $i \in Z_+$, $x_{0,j} \in R_+^n$, $j \in Z_+$ and every input sequence $u_{i,j} \in R_+^m$, $i,j \in Z_+$.

Theorem 5 [8]. The general model (9) is positive if and only if

$$\begin{aligned} A_k \in R_+^{n\times n}, \quad B_k \in R_+^{n\times m}, \quad k = 0,1,2, \\ C \in R_+^{p\times n}, \quad D \in R_+^{p\times m}. \end{aligned} \tag{11}$$

Substituting in (9a) $B_1 = B_2 = 0$ and $B_0 = B$ we obtain the first Fornasini-Marchesini model (FF-MM) and substituting in (9a) $A_0 = 0$ and $B_0 = 0$ we obtain the second Fornasini-Marchesini model (SF-MM).

The Roesser model of 2D linear systems has the form

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u_{i,j}, \tag{12a}$$

$$y_{i,j} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix} + Du_{i,j} \quad i,j \in Z_+, \tag{12b}$$

where $x_{i,j}^h \in R^{n_1}$ and $x_{i,j}^v \in R^{n_2}$ are the horizontal and vertical state vectors at the point (i,j), $u_{i,j} \in R^m$ and $y_{i,j} \in R^p$ are the input and output vectors and $A_{kl} \in R^{n_k\times n_l}$, $k,l = 1,2$, $B_{11} \in R^{n_1\times m}$, $B_{22} \in R^{n_2\times m}$, $C_1 \in R^{p\times n_1}$, $C_2 \in R^{p\times n_1}$, $D \in R^{p\times m}$.

Boundary conditions for (12a) have the form

$$x_{0,j}^h \in R^{n_1}, \, j \in Z_+ \quad \text{and} \quad x_{i,0}^v \in R^{n_2}, \, i \in Z_+. \tag{13}$$

The model (12) is called (internally) positive Roesser model if $x_{i,j}^h \in R_+^{n_1}$, $x_{i,j}^v \in R_+^{n_2}$ and $y_{i,j} \in R_+^p$, $i,j \in Z_+$ for any nonnegative boundary conditions

$$x_{0,j}^h \in R_+^{n_1}, \quad j \in Z_+, \quad x_{i,0}^v \in R_+^{n_2}, \quad i \in Z_+ \tag{14}$$

and all input sequences $u_{i,j}^{\in} R_+^m$, $i,j \in Z_+$.

Theorem 6 [8]. The Roesser model is positive if and only if

$$\begin{aligned} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in R_+^{n\times n}, \quad \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} \in R_+^{n\times m}, \\ \begin{bmatrix} C_1 & C_2 \end{bmatrix} \in R_+^{p\times n}, \quad D \in R_+^{p\times m}, \quad n = n_1 + n_2. \end{aligned} \tag{15}$$

Defining

$$x_{i,j=} \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \tag{16}$$

$$B_1 = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}$$

we may write the Roesser model in the form of SF-MM

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \tag{17}$$

3. Asymptotic stability of the positive 2D systems without delays

The positive general model (9a) is called asymptotically stable if for any bounded boundary conditions $x_{i,0} \in R_+^n$, $i \in Z_+$, $x_{0,j} \in R_+^n$, $j \in Z_+$ and zero inputs $u_{i,j} = 0$, $i,j \in Z_+$,

$$\lim_{i,j\to\infty} x_{i,j} = 0 \quad \text{for all} \quad x_{i,0} \in R_+^n, \; x_{0,j} \in R_+^n, \; i,j \in Z_+. \tag{18}$$

Theorem 7 [21]. For the positive general model (9) the following statements are equivalent:

1. The positive general model (9) is asymptotically stable
2.

$$\begin{aligned} \det\left(I_n - A_0 z_1 z_2 - A_1 z_2 - A_2 z_1\right) \neq 0 \\ \text{for} \quad \forall (z_1,z_2) \in \{(z_1,z_2) : |z_1| \leq 1, \, |z_2| \leq 1\}, \end{aligned} \tag{19}$$

3. There exists a strictly positive vector $\lambda \in R_+^n$ such that
$$[A_0 + A_1 + A_2 - I_n]\,\lambda << 0, \tag{20}$$
4. The sum of entries of every row (column) of the adjoint matrix $Adj\,[I_n - A_0 - A_1 - A_2]$ is strictly positive, i.e.
$$\begin{aligned}&(Adj\,[I_n - A_0 - A_1 - A_2])\mathbf{1}_n >> 0,\\ &(\mathbf{1}_n^T Adj\,[I_n - A_0 - A_1 - A_2] >> 0),\end{aligned} \tag{21}$$
where $\mathbf{1}_n^T = [1...1]$ and T denotes the transpose.
5. The positive 1D system
$$x_{i+1} = \begin{bmatrix} A_1 + A_2 & A_0 \\ I_n & 0 \end{bmatrix} x_i \quad i \in Z_+ \tag{22}$$
is asymptotically stable.

In particular case for $A_0 = 0$ we have the following corollary.

Corollary 1. [21]. The positive 2D SF-MM (17) is asymptotically stable if and only if the positive 1D system
$$x_{i+1} = (A_1 + A_2)\,x_i, \quad i \in Z_+, \tag{23}$$
is asymptotically stable.

Corollary 2. [21]. The positive 2D Roesser model (12) is asymptotically stable if and only if the positive 1D system
$$x_{i+1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x_i, \quad i \in Z_+, \tag{24}$$
is asymptotically stable.

4. Asymptotic stability of the positive 2D systems with delays

4.1. 2D Roesser model with delays. Consider the autonomous positive 2D Roesser model with q delays in state
$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \sum_{k=0}^{q} A_k \begin{bmatrix} x_{i-k,j}^h \\ x_{i,j-k}^v \end{bmatrix}, \quad i,j \in Z_+, \tag{25}$$
where $x_{i,j}^h \in R_+^{n_1}$ and $x_{i,j}^v \in R_+^{n_2}$ are the horizontal and vertical state vectors at the point (i,j) and
$$A_k = \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix}, \quad k = 0,1,\ldots,q. \tag{26}$$
Defining the vectors
$$\overline{x}_{i,j}^h = \begin{bmatrix} x_{i,j}^h \\ x_{i-1,j}^h \\ \vdots \\ x_{i-q,j}^h \end{bmatrix}, \quad \overline{x}_{i,j}^v = \begin{bmatrix} x_{i,j}^v \\ x_{i,j-1}^v \\ \vdots \\ x_{i,j-q}^v \end{bmatrix}, \tag{27}$$
we can write (25) in the form
$$\begin{bmatrix} \overline{x}_{i+1,j}^h \\ \overline{x}_{i,j+1}^v \end{bmatrix} = A \begin{bmatrix} \overline{x}_{i,j}^h \\ \overline{x}_{i,j}^v \end{bmatrix}, \quad i,j \in Z_+, \tag{28}$$
where
$$A = \left[\begin{array}{ccccc|ccccc} A_{11}^0 & A_{11}^1 & \cdots & A_{11}^{q-1} & A_{11}^q & A_{12}^0 & A_{12}^1 & \cdots & A_{12}^{q-1} & A_{12}^q \\ I_{n_1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I_{n_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline A_{21}^0 & A_{21}^1 & \cdots & A_{21}^{q-1} & A_{21}^q & A_{22}^0 & A_{22}^1 & \cdots & A_{22}^{q-1} & A_{22}^q \\ 0 & 0 & \cdots & 0 & 0 & I_{n_2} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I_{n_2} & 0 \end{array}\right] \in R^{N \times N}$$
$$N = (q+1)(n_1 + n_2) \tag{29}$$

Therefore, the 2D Roesser model with q delays (25) has been reduced to the 2D Roesser model without delays but with higher dimension. Applying to the model (28) Theorem 6 we obtain the following theorem.

Theorem 8. The 2D Roesser model with q delays (25) is positive if and only if $A_k \in R_+^{(n_1+n_2)\times(n_1+n_2)}$ for $k = 0,1,\ldots,q$ or equivalently $A \in R_+^{N \times N}$.

From Corollary 2 applied to the model (28) we have the following theorem.

Theorem 9. The positive 2D Roesser with q delays (25) is asymptotically stable if and only if the positive 1D system
$$x_{i+1} = Ax_i, \quad i \in Z_+, \tag{30}$$
with the matrix A defined by (29) is asymptotically stable.

To check the asymptotic stability of the positive 2D system (25) we may use any of the conditions of Theorems 2 and 3.

Example 1. Check the asymptotic stability of the positive 2D Roesser model (25) for $q = 1$ and with the matrices
$$\begin{aligned} A_0 &= \left[\begin{array}{cc|c} 0.1 & 0.2 & 0 \\ 0 & 0.1 & 0.3 \\ \hline 0 & 0 & 0.2 \end{array}\right], \\ A_1 &= \left[\begin{array}{cc|c} 0.2 & 0.1 & 0.2 \\ 0 & 0.1 & 0.2 \\ \hline 0 & 0 & 0.2 \end{array}\right]. \end{aligned} \tag{31}$$

To verify the asymptotic stability we shall use the condition 2) of the Theorem 3. The matrix (7) in this case has the form
$$\overline{A} = I_n - A = \begin{bmatrix} I_2 - A_{11}^0 & -A_{11}^1 & -A_{12}^0 & -A_{12}^1 \\ -I_2 & I_2 & 0 & 0 \\ -A_{21}^0 & -A_{21}^1 & I_1 - A_{22}^0 & -A_{22}^1 \\ 0 & 0 & -I_1 & I_1 \end{bmatrix} =$$
$$= \begin{bmatrix} 0.9 & -0.2 & -0.2 & -0.1 & 0 & -0.2 \\ 0 & 0.9 & 0 & -0.1 & -0.3 & -0.2 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & -0.2 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{32}$$

The principal minors of the matrix (32) are positive since

$$M_1 = 0.9,$$

$$M_2 = \begin{vmatrix} 0.9 & -0.2 \\ 0 & 0.9 \end{vmatrix} = 0.81,$$

$$M_3 = \begin{vmatrix} 0.9 & -0.2 & -0.2 \\ 0 & 0.9 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0.63,$$

$$M_4 = \begin{vmatrix} 0.9 & -0.2 & -0.2 & -0.1 \\ 0 & 0.9 & 0 & -0.1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix} = 0.7,$$

$$M_5 = 0.8 M_4,$$
$$M_6 = 0.6 M_4.$$

Therefore, the condition 2) of Theorem 3 is met and the positive Roesser model (25) with (31) is asymptotically stable.

The same result we obtain using other conditions of Theorems 2 and 3.

In a similar way the considerations can be extended for the positive 2D Roesser model of the form

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \sum_{k=0}^{q_1} \sum_{l=0}^{q_2} A_{kl} \begin{bmatrix} x^h_{i-k,j-l} \\ x^v_{i-k,j-l} \end{bmatrix}, \quad i,j \in Z_+, \tag{33}$$

where $x^h_{i,j} \in R^{n_1}$ and $x^v_{i,j} \in R^{n_2}$ are the horizontal and vertical state vectors at the point (i,j) and $A_{kl} \in R^{(n_1+n_2)\times(n_1+n_2)}$.

4.2. 2D general model with delays. Consider the autonomous positive 2D general model with q delays in state

$$x_{i+1,j+1} = \sum_{k=0}^{q} \left(A^0_k x_{i-k,j-k} + A^1_k x_{i+1-k,j-k} + A^2_k x_{i-k,j+1-k} \right) \quad i,j \in Z_+, \tag{34}$$

where $x_{i,j} \in R^n_+$ is the state vector at the point (i,j) and $A^t_k \in R^{n\times n}_+$, $k = 0,1,\dots,q$; $t = 0,1,2$. Defining the vector

$$\overline{x}_{i,j} = \begin{bmatrix} x_{i,j} \\ x_{i-1,j-1} \\ x_{i-2,j-2} \\ \vdots \\ x_{i-q,j-q} \end{bmatrix} \in R^{\overline{N}}, \quad \overline{N} = (q+1)n \tag{35a}$$

and the matrices

$$\overline{A}_0 = \begin{bmatrix} A^0_0 & A^0_1 & \cdots & A^0_{q-1} & A^0_q \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_n & 0 \end{bmatrix},$$

$$\overline{A}_1 = \begin{bmatrix} A^1_0 & A^1_1 & \cdots & A^1_{q-1} & A^1_q \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{35b}$$

$$\overline{A}_2 = \begin{bmatrix} A^2_0 & A^2_1 & \cdots & A^2_{q-1} & A^2_q \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

we can write (34) in the form

$$\overline{x}_{i+1,j+1} = \overline{A}_0 \overline{x}_{i,j} + \overline{A}_1 \overline{x}_{i+1,j} + \overline{A}_2 \overline{x}_{i,j+1} \quad i,j \in Z_+. \tag{36}$$

Therefore, the general model with q delays (34) has been reduced to the 2D general model without delays but with higher dimension. Applying to the model (36) Theorem 5 we obtain the following.

Theorem 10. The general 2D model with q delays (34) is positive if and only if $A^t_k \in R^{n\times n}_+$ for $t = 0,1,2$ and $k = 0,1,\dots,q$ or equivalently $\overline{A}_t \in R^{\overline{N}\times\overline{N}}_+$ for $t = 0,1,2$.

From the condition 5) of Theorem 7 applied to the model (36) we have the following theorem.

Theorem 11. The positive 2D general model with q delays (34) is asymptotically stable if and only if the positive 1D system

$$x_{i+1} = \begin{bmatrix} \overline{A}_1 + \overline{A}_2 & \overline{A}_0 \\ I_{\overline{N}} & 0 \end{bmatrix} x_i, \quad i \in Z_+, \tag{37}$$

is asymptotically stable, where the matrices $\overline{A}_t$ for $t = 0,1,2$ are defined by (35b).

To check the asymptotic stability of the positive 2D general model with delays (34) we may use any of the conditions of Theorems 2 and 3.

For $\overline{A}_0 = 0$ from Theorem 11 we have the following corollary.

Corollary 3.

The positive 2D SF-MM with q delays (34) is asymptotically stable if and only if the positive 1D system

$$\overline{x}_{i+1} = \left[\overline{A}_1 + \overline{A}_2\right] \overline{x}_i, \quad i \in Z_+, \tag{38}$$

is asymptotically stable.

Example 2. Consider the positive model (34) for $q = 1$ with the matrices

$$A_0^0 = \begin{bmatrix} a & 0.1 \\ 0 & 0.2 \end{bmatrix}, \quad A_0^1 = \begin{bmatrix} 0.1 & 0.2 \\ 0 & b \end{bmatrix},$$
$$A_0^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{39}$$
$$A_1^1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \quad A_1^2 = \begin{bmatrix} 0 & 0.2 \\ 0 & c \end{bmatrix}$$
$$(a, b, c \geq 0)$$

Find the values of the coefficients a, b, c for which the positive model is asymptotically stable.

Using (35b) we obtain

$$\overline{A}_0 = \begin{bmatrix} A_0^0 & A_1^0 \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} a & 0.1 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\overline{A}_1 = \begin{bmatrix} A_0^1 & A_1^1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0.1 \\ 0 & b & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{A}_2 = \begin{bmatrix} A_0^2 & A_1^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To stability analysis of the system we use Theorem 11.Taking into account that

$$I_{4n} - A = \begin{bmatrix} I_{2n} - \overline{A}_1 - \overline{A}_2 & -\overline{A}_0 \\ -I_{2n} & I_{2n} \end{bmatrix} =$$

$$= \begin{bmatrix} 0.9 & -0.2 & -0.2 & -0.3 & -a & -0.1 & 0 & 0 \\ 0 & 1-b & 0 & -(0.1+c) & 0 & -0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and using the condition 2) of Theorem 3 we obtain

$$M_1 = 0.9, \quad M_2 = \begin{vmatrix} 0.9 & -0.2 \\ 0 & 1-b \end{vmatrix} = 0.9(1-b) > 0,$$

$$M_3 = \begin{vmatrix} 0.9 & -0.2 & -0.2 \\ 0 & 1-b & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0.9(1-b) > 0,$$

$$M_4 = \begin{vmatrix} 0.9 & -0.2 & -0.2 & -0.3 \\ 0 & 1-b & 0 & -(0.1+c) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0.9(1-b) > 0,$$

$$M_5 = \begin{vmatrix} 0.9 & -0.2 & -0.2 & -0.3 & -a \\ 0 & 1-b & 0 & -(0.1+c) & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{vmatrix} = (0.7-a)(1-b) > 0,$$

$$M_6 = \begin{vmatrix} 0.9 & -0.2 & -0.2 & -0.3 & -a & -0.1 \\ 0 & 1-b & 0 & -(0.1+c) & 0 & -0.2 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{vmatrix} = (0.7-a)\left[0.7-(b+c)\right] > 0,$$

$$M_8 = M_7 = M_6.$$

Therefore, the positive model (34) with (39) is asymptotically stable if and only if $0 \leq a < 0.7$ and $0 \leq b + c < 0.7$.

In a similar way the considerations can be extended for the positive 2D general model of the form

$$x_{i+1,j+1} = \sum_{k=0}^{q_1}\sum_{l=0}^{q_2}\left(A_{kl}^0 x_{i-k,j-l} + A_{kl}^1 x_{i-k+1,j-l} + A_{kl}^2 x_{i-k,j-l+1}\right), \quad i,j \in Z_+, \tag{40}$$

where $x_{i,j} \in R_+^n$ is the state vector at the point (i,j) and $A_{kl}^t \in R_+^{n\times n}$, $t = 0,1,2$ and $k = 0,1,\dots,q_1$; $l = 0,1,\dots,q_2$.

5. Concluding remarks

The asymptotic stability of positive 2D linear systems with delays described by the general model, Fornasini-Marchesini models and Roesser model has been addressed. New necessary and sufficient conditions for the asymptotic stability of positive 2D linear systems with delays have been established. It has been shown that the checking of the asymptotic stability of positive 2D linear systems with delays can be reduced to the checking of the asymptotic stability of corresponding positive 1D linear systems without delays. The efficiency of the new criterions has been demonstrated on two numerical examples.

An extension of these considerations for 2D positive continuous-time linear systems is an open problem.

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