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On the one-dimensional wave propagation in inhomogeneous elastic layer

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Abstract. The standard approach to the wave propagation in an inhomogeneous elastic layer leads to the displacement in a form of a product of a function of space and a harmonic function of time. This product represents the standing, and not the running wave. The part depending on the space variable is governed by the linear ordinary second order differential equation. In order to calculate the propagation speed in the present paper the inhomogeneous material is separated by a plane into two parts. Between the two inhomogeneous parts the virtual homogeneous elastic extra layer is added. The elasticity modulus and the mass density of the extra layer have the same values as the inhomogeneous material on the separation plane. In further calculations the extra layer is assumed to be infinitesimally thin. The virtual layer allows to decompose the motion into two waves: a wave running to the right and a wave running to the left. Energy conservation equation is derived.

1. Introduction

The one-dimensional time-dependent displacement in the homogeneous linear elastic layer is expressed by a simple formula. In this formula the propagation speed is explicitly present and the motion of the discontinuity surface separating the disturbed and undisturbed regions may be easily defined. In contrast to this the expression for a displacement in the inhomogeneous elastic layer has entirely different form in which the propagation speed is not explicitly present. In connection with this fact the analysis of wave propagation in inhomogeneous layer demands special treatment. In the present paper such treatment is proposed and discussed. Preliminary analysis of the problem has been given in [1]. Standard approach in acoustics is based on replacement the inhomogeneous material by material piecewise homogeneous, cf. e.g. [2].

2. Inhomogeneous region

The inhomogeneous elastic layer of density ρ(x) and elastic modulus $E(x)$ is situated between $x = 0$ and $x = h$, where h is fixed. Consider a one-dimensional time-dependent motion of this layer. The longitudinal displacement u is the function of time t and spatial variable x, $u = u(x,t)$. The function $u(x,t)$ satisfies the linear differential equation of motion

$$\frac{\partial}{\partial x}\left[E(x)\frac{\partial u(x,t)}{\partial x}\right] = \rho(x)\frac{\partial^2 u(x,t)}{\partial t^2}. \tag{1}$$

It is assumed, that $E(x)$ and $\rho(x)$ for each x are not equal zero. The displacement is expected in the form

$$u(x,t) = v(x)\exp i\omega t. \tag{2}$$

Here ω denotes the circular frequency. The actual displacement is the real part of the complex-valued $u(x,t)$. In accord with the partial differential equation (1) the function $v(x)$ satisfies the ordinary differential equation

$$\frac{d}{dx}\left[E(x)\frac{dv(x)}{dx}\right] + \omega^2\rho(x)v(x) = 0. \tag{3}$$

Denote the two linearly independent real solutions of (3) by $\alpha(x)$ and $\beta(x)$, respectively. They satisfy the differential equations

$$\begin{aligned}\frac{d}{dx}\left[E(x)\frac{d\alpha(x)}{dx}\right] + \omega^2\rho(x)\alpha(x) = 0,\\ \frac{d}{dx}\left[E(x)\frac{d\beta(x)}{dx}\right] + \omega^2\rho(x)\beta(x) = 0.\end{aligned} \tag{4}$$

Therefore in the inhomogeneous material (for $0 < x < h$) the displacement $u(x,t)$ is given by the expression

$$u(x,t) = A\alpha(x)\exp i\omega t + B\beta(x)\exp i\omega t, \tag{5}$$

where A and B are complex-valued constants. Note that the wave speeds are not explicitly present in this expression.

Assume that at the left and the right side of the inhomogeneous region two different homogeneous regions are situated, cf Fig. 1.

In the homogeneous region for $x < 0$ the elastic modulus and the density are constant $\rho(x) = \rho_0 = const$, $E(x) = E_0 = const$. In the homogeneous region for $x > h$ there is $\rho(x) = \rho_2 = const$, $E(x) = E_2 = const$. On the boundaries $x = 0$ and $x = h$ the elastic modulus and the density are assumed to have no jump. The equations of motion for the homogeneous regions are respectively

$$E_0\frac{\partial^2 u(x,t)}{\partial x^2} = \rho_0\frac{\partial^2 u(x,t)}{\partial t^2} \;\text{ for } \text{x} < 0, \tag{6}$$

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$$E_2 \frac{\partial^2 u(x,t)}{\partial x^2} = \rho_2 \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{for x > h}, \tag{7}$$

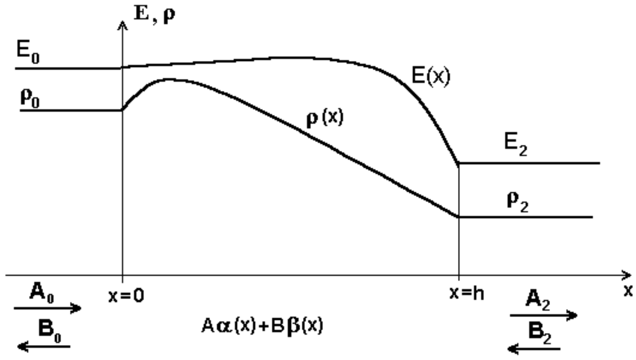

Fig. 1. Homogeneous and inhomogeneous regions

The following complex-valued displacements $u(x,t)$

$$u(x,t) = A_0 \exp i\omega \left(t - \frac{x}{c_0}\right) + B_0 \exp i\omega \left(t + \frac{x}{c_0}\right), \tag{8}$$

$$u(x,t) = A_2 \exp i\omega \left(t - \frac{x-h}{c_2}\right) + B_2 \exp i\omega \left(t + \frac{x-h}{c_2}\right) \tag{9}$$

satisfy the equations of motion in the homogeneous regions for $x < 0$ and $x > h$, respectively. In (8) and (9)

$$c_0 = \sqrt{\frac{E_0}{\rho_0}}, \quad c_2 = \sqrt{\frac{E_2}{\rho_2}}, \tag{10}$$

are the propagation speeds in the homogeneous regions $x < 0$ and $x > h$, respectively. The actual displacements equals the real part of the complex-valued functions $u(x,t)$, as given by (8) and (9). The expressions in (8) and (9) have the form of sinusoidal waves propagating to the left and to the right. The term proportional to A_0 represents the incident wave coming from the left, and the term proportional to B_2 represents the incident wave coming from the right. The term proportional to A_2 represents the transmitted wave running to the right, and the term proportional to B_0 represents the transmitted wave running to the left. Further we assume that A_0 and B_0 are given in advance. The propagation speeds c_0 and c_2, are explicitly present in the formula for the displacement.

In each region the stress equals $E(x)\partial u(x,t)/\partial x$. The actual stress is the real part of this complex-valued expression. At the separation plane $x = 0$ both the displacement u and the stress must be continuous. Since the values of the elastic modulus $E(x)$ have the same value on both sides of $x = 0$, the continuity relations reduce to

$$\begin{aligned} A_0 + B_0 &= A\alpha(0) + B\beta(0), \\ -\frac{i\omega}{c_0} A_0 + \frac{i\omega}{c_0} B_0 &= A \frac{d\alpha(0)}{dx} + B \frac{d\beta(0)}{dx}. \end{aligned} \tag{11}$$

The above relations allow to express the amplitudes (A, B) in the inhomogeneous region by the amplitudes (A_0, B_0). There is

$$\begin{aligned} A &= \frac{1}{M_0}\left\{+\left[\beta_0' + \frac{i\omega}{c_0}\beta_0\right] A_0 + \left[\beta_0' - \frac{i\omega}{c_0}\beta_0\right] B_0\right\}, \\ B &= \frac{1}{M_0}\left\{-\left[\alpha_0' + \frac{i\omega}{c_0}\alpha_0\right] A_0 - \left[\alpha_0' - \frac{i\omega}{c_0}\alpha_0\right] B_0\right\}, \end{aligned} \tag{12}$$

where

$$M_0 = \alpha_0 \beta_0' - \beta_0 \alpha_0', \tag{13}$$

$$\alpha_0 = \alpha(0), \ \beta_0 = \beta(0), \ \alpha_0' = \frac{d\alpha(0)}{dx}, \ \beta_0' = \frac{d\beta(0)}{dx}. \tag{14}$$

Since $\alpha(x)$ and $\beta(x)$ are linearly independent in general there is $M_0 \neq 0$. The case $M_0 = 0$ demands special treatment.

In accord with (5) for the inhomogeneous region $0 < x < h$ the displacement is given by the expression

$$\begin{aligned} u(x,t) = &\frac{1}{M_0}\left\{+\left[\beta_0' + \frac{i\omega}{c_0}\beta_0\right] A_0 + \left[\beta_0' - \frac{i\omega}{c_0}\beta_0\right] B_0\right\} \\ &\alpha(x) \exp i\omega t, \\ &+\frac{1}{M_0}\left\{-\left[\alpha_0' + \frac{i\omega}{c_0}\alpha_0\right] A_0 - \left[\alpha_0' - \frac{i\omega}{c_0}\alpha_0\right] B_0\right\} \\ &\beta(x) \exp i\omega t. \end{aligned} \tag{15}$$

The constants A_2 and B_2 present in the expression (9) for the displacement in the homogeneous region $x > h$ may be expressed by A_0 and B_0 . The coresponding formulae are not necessary for further calculations, therefore they are not quoted here. The expression (15) does not expose the fact, that the displacement $u(x,t)$ is a propagating wave, or more exactly: sum of two sinusoidal propagating vaves. Their speeds and amplitudes are not known. Entirely different approach must be applied to the motion of propagating, time-dependent discontinuity surface. However in the present paper only the sinusoidal waves are treated.

The displacement (15) has a form of a product of two functions: a function of spatial variable x and a function of time t, therefore a form of a standing wave. On the other hand, a possibility of separation of the motion into a wave running to the right and a wave running to the left in each material, at least for some inhomogeneities, is evident. The propagation speed c for $0 < x < h$ must be a function of x, $c = c(x)$. This speed is not present in (15). Deriving a separation valid for the inhomogeneous material is the purpose of the further analysis in the present paper. Note that for the homogeneous material the separation is trivial.

3. Virtual homogeneous layer

Consider now the more involved case, when without changing the properties of the material the homogeneous layer of thickness d is added at $x = s$. This is a virtual layer not present in the real system. Its thickness d is arbitrary, and in particular case d may tend to zero. It is assumed that the density of the virtual layer coincides with the density of the inhomogeneous layer at $x = s$. Similarly the elastic modulus of the virtual layer coincides with the elastic modulus of the inhomogeneous layer at $x = s$. The only purpose of adding the virtual layer is to make possible the discussion of wave propagation in the inhomogeneous region and to calculate the propagation speed $c(x)$ in the inhomogeneous layer.

The extra layer divides the inhomogeneous region into two parts, cf. Fig. 2. All amplitudes corresponding to the above situation have the superscript *. In the homogeneous region $x < 0$ the density $\rho(x)$ and the elastic modulus $E(x)$ are constant. There propagate two waves expressed by the two terms of the relation

$$u = A_0^* \exp i\omega \left(t - \frac{x-s}{c_0}\right) + B_0^* \exp i\omega \left(t + \frac{x-s}{c_0}\right). \tag{16}$$

In general the two amplitudes A_0^* and B_0^* are complex valued. For convenience in the expression was written x-s instead of x. This is equivalent to a change of the amplitudes. In the inhomogeneous region $0 < x < s$ the displacement is given by the formula

$$u = A_L^* \alpha(x) \exp i\omega t + B_L^* \beta(x) \exp i\omega t, \tag{17}$$

where the functions $\alpha(x)$ and $\beta(x)$ are the two linearly independent real solutions satisfying the differential equation (3), and A_L^* and B_L^* are two arbitrary complex constants. The subscript L has been added, since the amplitudes are in general different from A, B used in Chapter 1. In the region $s + d < x < h + d$ the amplitudes will have the values A_R^* and B_R^* differing from A_L^* and B_L^* and from A, B.

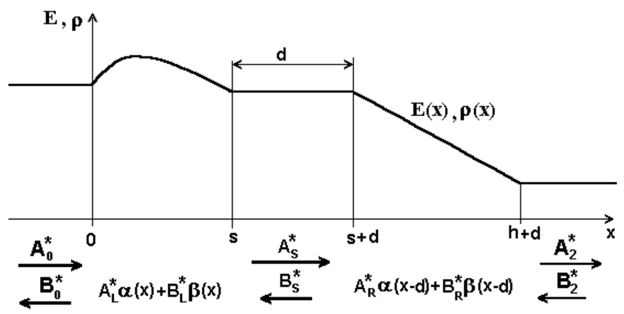

Fig. 2. System with the virtual homogeneous layer

Since on the plane $x = 0$ both the displacement and the stress are continuous at $x = 0$ the amplitudes must satisfy the algebraic relations

$$\begin{aligned} A_L^* \alpha(0) + B_L^* \beta(0) &= A_0^* + B_0^*, \\ A_L^* \alpha'(0) + B_L^* \beta'(0) &= -\frac{i\omega}{c_0} A_0^* + \frac{i\omega}{c_0} B_0^*. \end{aligned} \tag{18}$$

Note that the two linearly independent solutions $\alpha(x)$ and $\beta(x)$ of (4) for each inhomogeneity and given frequency ω always exist. For each case their numerical approximation may be calculated, for some particular inhomogeneities the analytical forms of $\alpha(x)$ and $\beta(x)$ are known. If these forms are known, then the amplitudes of the waves in the extra layer are known functions of the amplitudes of the waves in the region $x < 0$[1]. Using the shorthand notation (14) the following expressions are obtained

$$\begin{aligned} A_L^* &= \frac{1}{M_0} \left\{ + \left[\beta_0' + \frac{i\omega}{c_0}\beta_0\right] A_0^* + \left[\beta_0' - \frac{i\omega}{c_0}\beta_0\right] B_0^* \right\}, \\ B_L^* &= \frac{1}{M_0} \left\{ - \left[\alpha_0' + \frac{i\omega}{c_0}\alpha_0\right] A_0^* - \left[\alpha_0' - \frac{i\omega}{c_0}\alpha_0\right] B_0^* \right\}. \end{aligned} \tag{19}$$

The parameter M_0 is defined by (13).

The virtual layer $s < x < x + d$ is homogeneous. There propagate two waves with known propagation speed c_s, which is the propagation speed in the virtual layer. The first wave is running to the right, the second to the left. The displacement is

$$u = A_S^* \exp i\omega \left(t - \frac{x-s}{c_S}\right) + B_S^* \exp i\omega \left(t + \frac{x-s}{c_S}\right). \tag{20}$$

$$c_S = \sqrt{\frac{E(s)}{\rho(s)}}. \tag{21}$$

The complex-valued A_S^*, B_S^* are the amplitudes of both waves, respectively. Concentrate on the separation plane $x = s$. On this plane both the displacement and the stress are continuous. Since the elasticity modulus $E(x)$ has the same value at both sides of $x = s$ the amplitudes satisfy the algebraic equations

$$\begin{aligned} A_L^* \alpha(s) + B_L^*(s) &= A_S^* + B_S^*, \\ A_L^* \alpha'(s) + B_L^* \beta'(s) &= -\frac{i\omega}{c_S} A_S^* + \frac{i\omega}{c_S} B_S^*. \end{aligned} \tag{22}$$

The above equations allow to express the amplitudes A_S^*, B_S^* by the amplitudes A_L^*, B_L^*

$$\begin{aligned} 2A_S^* &= \left[\alpha_S - \frac{c_S}{i\omega}\alpha_S'\right] A_L^* + \left[\beta_S - \frac{c_S}{i\omega}\beta_S'\right] B_L^*, \\ 2B_S^* &= \left[\alpha_S + \frac{c_S}{i\omega}\alpha_S'\right] A_L^* + \left[\beta_S + \frac{c_S}{i\omega}\beta_S'\right] B_L^*. \end{aligned} \tag{23}$$

Taking into account the formulae (19) and chaining the results the following expressions for the amplitudes are obtained

[1]E.g. the inhomogeneity $E(x) = \frac{1}{x^2+1}$, $\omega^2\rho(x) = \frac{2}{(x^2+1)^2}$ leads to the equation $\frac{d^2v(x)}{dx^2} - \frac{2x}{x^2+1}\frac{dv(x)}{dx} + \frac{2}{x^2+1}v(x) = 0$ quoted in [3] as Eq. 2.227. Its two independent solutions are $\alpha(x) = x$, $\beta(x) = x^2 - 1$and functions $\alpha(0)$, $\beta(0)$, $\alpha'(0)$, $\beta'(0)$ in (18) are known.

$$
\begin{aligned}
2M_0A_S^* &= \Big\{-[\alpha_0'\beta_S-\beta_0'\alpha_S]+\frac{c_S}{i\omega}[\alpha_0'\beta_S'-\beta_0'\alpha_S'] \\
&\quad -\frac{i\omega}{c_0}[\alpha_0\beta_S-\beta_0\alpha_S]+\frac{c_S}{c_0}[\alpha_0\beta_S'-\beta_0\alpha_S']\Big\}A_0^* \\
&\quad +\Big\{-[\alpha_0'\beta_S-\beta_0'\alpha_S]+\frac{c_S}{i\omega}[\alpha_0'\beta_S'-\beta_0'\alpha_S'] \\
&\quad +\frac{i\omega}{c_0}[\alpha_0\beta_S-\beta_0\alpha_S]-\frac{c_S}{c_0}[\alpha_0\beta_S'-\beta_0\alpha_S']\Big\}B_0^*, \\
2M_0B_S^* &= \Big\{-[\alpha_0'\beta_S-\beta_0'\alpha_S]-\frac{c_S}{i\omega}[\alpha_0'\beta_S'-\beta_0'\alpha_S'] \\
&\quad -\frac{i\omega}{c_0}[\alpha_0\beta_S-\beta_0\alpha_S]-\frac{c_S}{c_0}[\alpha_0\beta_S'-\beta_0\alpha_S']\Big\}A_0^* \\
&\quad +\Big\{-[\alpha_0'\beta_S-\beta_0'\alpha_S]-\frac{c_S}{i\omega}[\alpha_0'\beta_S'-\beta_0'\alpha_S'] \\
&\quad +\frac{i\omega}{c_0}[\alpha_0\beta_S-\beta_0\alpha_S]+\frac{c_S}{c_0}[\alpha_0\beta_S'-\beta_0\alpha_S']\Big\}B_0^*.
\end{aligned} \tag{24}
$$

The complex-valued A_S^*, B_S^* have been expressed by the the amplitudes of the waves running in the homogeneous region $x<0$. Note that the formulae are valid for any thickness d.

The expression (20) represents the waves in the homogeneous region $s<x<s+d$. The wave of amplitude A_S^* is running to the right, and the wave of amplitude B_S^* is running to the left. They have the same speeds, defined by the relation (21). It is obvious, that the amplitudes depend on the amplitudes in the region $x<0$. Note, that the expressions for the amplitudes A_S^*, B_S^* contain the values of the functions $\alpha(x)$ and $\beta(x)$ and their derivatives $\alpha'(x)$ and $\beta'(x)$ at $x=s$.

The actual displacement equals the real part of the complex-valued $u(x,t)$. Pass to the calculation of the real products $A_S^*\overline{A_S^*}$, $B_S^*\overline{B_S^*}$, equal to the squared real amplitudes. In accord with (23) and (24) we have

$$
\begin{aligned}
&4M_0^2A_S^*\overline{A_S^*}= \\
&\Big\{\Big[(\alpha_0'\beta_S-\beta_0'\alpha_S)-\frac{c_S}{c_0}(\alpha_0\beta_S'-\beta_0\alpha_S')\Big]^2 \\
&+\Big[\frac{\omega}{c_0}(\alpha_0\beta_S-\beta_0\alpha_S)+\frac{c_S}{\omega}(\alpha_0'\beta_S'-\beta_0'\alpha_S')\Big]^2\Big\}A_0^*\overline{A_0^*} \\
&+\Big\{\Big[(\alpha_0'\beta_S-\beta_0'\alpha_S)+i\frac{\omega}{c_0}(\alpha_0\beta_S-\beta_0\alpha_S)\Big]^2 \\
&+\Big[\frac{c_S}{\omega}(\alpha_0'\beta_S'-\beta_0'\alpha_S')+i\frac{c_S}{c_0}(\alpha_0\beta_S'-\beta_0\alpha_S')\Big]^2\Big\}A_0^*\overline{B_0^*} \\
&+\Big\{\Big[(\alpha_0'\beta_S-\beta_0'\alpha_S)-i\frac{\omega}{c_0}(\alpha_0\beta_S-\beta_0\alpha_S)\Big]^2 \\
&+\Big[\frac{c_S}{\omega}(\alpha_0'\beta_S'-\beta_0'\alpha_S')-i\frac{c_S}{c_0}(\alpha_0\beta_S'-\beta_0\alpha_S')\Big]^2\Big\}\overline{A_0^*}B_0^* \\
&+\Big\{\Big[(\alpha_0'\beta_S-\beta_0'\alpha_S)+\frac{c_S}{c_0}(\alpha_0\beta_S'-\beta_0\alpha_S')\Big]^2 \\
&+\Big[\frac{\omega}{c_0}(\alpha_0\beta_S-\beta_0\alpha_S)-\frac{c_S}{\omega}(\alpha_0'\beta_S'-\beta_0'\alpha_S')\Big]^2\Big\}B_0^*\overline{B_0^*}^2,
\end{aligned}
$$

$$
\begin{aligned}
&4M_0^2B_S^*\overline{B_S^*}= \\
&\Big\{\Big[(\alpha_0'\beta_S-\beta_0'\alpha_S)-\frac{c_S}{c_0}(\alpha_0\beta_S'-\beta_0\alpha_S')\Big]^2 \\
&+\Big[\frac{\omega}{c_0}(\alpha_0\beta_S-\beta_0\alpha_S)+\frac{c_S}{\omega}(\alpha_0'\beta_S'-\beta_0'\alpha_S')\Big]^2\Big\}A_0^*\overline{A_0^*} \\
&+\Big\{\Big[(\alpha_0'\beta_S-\beta_0'\alpha_S)+i\frac{\omega}{c_0}(\alpha_0\beta_S-\beta_0\alpha_S)\Big]^2 \\
&+\Big[\frac{c_S}{\omega}(\alpha_0'\beta_S'-\beta_0'\alpha_S')+i\frac{c_S}{c_0}(\alpha_0\beta_S'-\beta_0\alpha_S')\Big]^2\Big\}A_0^*\overline{B_0^*} \\
&+\Big\{\Big[(\alpha_0'\beta_S-\beta_0'\alpha_S)-i\frac{\omega}{c_0}(\alpha_0\beta_S-\beta_0\alpha_S)\Big]^2 \\
&+\Big[\frac{c_S}{\omega}(\alpha_0'\beta_S'-\beta_0'\alpha_S')-i\frac{c_S}{c_0}(\alpha_0\beta_S'-\beta_0\alpha_S')\Big]^2\Big\}\overline{A_0^*}B_0^* \\
&+\Big\{\Big[(\alpha_0'\beta_S-\beta_0'\alpha_S)+\frac{c_S}{c_0}(\alpha_0\beta_S'-\beta_0\alpha_S')\Big]^2 \\
&+\Big[\frac{\omega}{c_0}(\alpha_0\beta_S-\beta_0\alpha_S)-\frac{c_S}{\omega}(\alpha_0'\beta_S'-\beta_0'\alpha_S')\Big]^2\Big\}B_0^*\overline{B_0^*}^2,
\end{aligned} \tag{25}
$$

There follows the identity

$$
\begin{aligned}
&M_0^2[A_S^*\overline{A_S^*}-B_S^*\overline{B_S^*}]= \\
&\frac{c_S}{c_0}\{-(\alpha_0\beta_S'-\beta_0\alpha_S')(\alpha_0'\beta_S-\beta_0'\alpha_S) \\
&+(\alpha_0'\beta_S'-\beta_0'\alpha_S')(\alpha_0\beta_S-\beta_0\alpha_S)\}[A_0^{*2}-B_0^{*2}].
\end{aligned} \tag{26}
$$

The expression in the bracket at the right-hand side may be simplified to the form

$$
\begin{aligned}
&-(\alpha_0\beta_S'-\beta_0\alpha_S')(\alpha_0'\beta_S-\beta_0'\alpha_S) \\
&+(\alpha_0'\beta_S'-\beta_0'\alpha_S')(\alpha_0\beta_S-\beta_0\alpha_S)=M_0M_S,
\end{aligned} \tag{27}
$$

where in analogy with (13)

$$
M_S\;=\alpha_S\beta_S'-\beta_S\alpha_S', \tag{28}
$$

was defined. Since $\alpha(x)$ and $\beta(x)$ are linearly independent in general there is $M_S\neq 0$. The singular cases $M_0=0$ and $M_S=0$ are not treated here. There follows the identity

$$
\frac{1}{c_SM_S}\left[A_S^*\overline{A_S^*}-B_S^*\overline{B_S^*}\right]=\frac{1}{c_0M_0}\left[A_0^*\overline{A_0^*}-B_0^*\overline{B_0^*}\right]. \tag{29}
$$

4. Energy flux

Define the function

$$
M(x)\;=\alpha(x)\frac{d\beta(x)}{dx}-\beta(x)\frac{d\alpha(x)}{dx}. \tag{30}
$$

Since $\alpha(x)$ and $\beta(x)$ are two linearly independent real solutions of a differential equation $M(x)$ is not identically zero. Obviously $M_0 = M(0)$ and $M_S = M(s)$. Consider the relation between $M(x)$ and $E(x)$ which is essential in further analysis. Differentiating (30) with respect to x we obtain

$$\frac{dM(x)}{dx} = \alpha(x)\frac{d^2\beta(x)}{dx^2} - \beta(x)\frac{d^2\alpha(x)}{dx^2}. \tag{31}$$

From (4) there follow the expressions for the second derivatives of $\alpha(x)$ and $\beta(x)$

$$\begin{aligned} \frac{d^2\alpha(x)}{dx^2} &= -\frac{1}{E(x)}\left[\frac{dE(x)}{dx}\frac{d\alpha(x)}{dx} + \omega^2\rho(x)\alpha(x)\right], \\ \frac{d^2\beta(x)}{dx^2} &= -\frac{1}{E(x)}\left[\frac{dE(x)}{dx}\frac{d\beta(x)}{dx} + \omega^2\rho(x)\beta(x)\right]. \end{aligned} \tag{32}$$

Substitute (32) into (31) to obtain

$$\begin{aligned} \frac{dM(x)}{dx} &= -\alpha(x)\frac{1}{E(x)}\frac{dE(x)}{dx}\frac{d\beta(x)}{dx} \\ +\beta(x)\frac{1}{E(x)}\frac{dE(x)}{dx}\frac{d\alpha(x)}{dx} &= -\frac{1}{E(x)}\frac{dE(x)}{dx}M(x). \end{aligned} \tag{33}$$

Finally

$$\frac{1}{M(x)}\frac{dM(x)}{dx} = -\frac{1}{E(x)}\frac{dE(x)}{dx}. \tag{34}$$

After integration of the above equation there follows

$$M(x) = \frac{1}{E(x)} \times \text{const.} \tag{35}$$

Therefore, the identity (27) may be written in the form

$$\frac{E_S}{c_S}\left[A_S^*\overline{A_S^*} - B_S^*\overline{B_S^*}\right] = \frac{E_0}{c_0}\left[A_0^*\overline{A_0^*} - B_0^*\overline{B_0^*}\right]. \tag{36}$$

Analyse the above relation. Denote by σ_{A0} and σ_{B0} the stresses corresponding respectively to the wave running to the right and to the wave running to the left in the region $x < 0$. The elastic energy densities L_{A0}, L_{B0} of unit cross-section, per unit length in space corresponding to the waves (8) running in the homogeneous region are given by the elementary formulae

$$L_{A0} = \frac{\sigma_{A0}\overline{\sigma_{A0}}}{2E_0}, \quad L_{B0} = \frac{\sigma_{A0}\overline{\sigma_{A0}}}{2E_0}, \tag{37}$$

where E_0 denotes the elastic modulus for the homogeneous region $x < 0$. The stress $\sigma(x)$ is proportional to the strain $\sigma(x) = E(x)\partial u/\partial x$. For the wave running to the right and the wave running to the left we have respectively

$$\begin{aligned} \sigma_{A0}(x,t) &= -E_0\frac{i\omega}{c_0}A_0\exp i\omega\left(t - \frac{x}{c_0}\right), \\ \sigma_{B0}(x,t) &= +E_0\frac{i\omega}{c_0}B_0\exp i\omega\left(t + \frac{x}{c_0}\right). \end{aligned} \tag{38}$$

Analogous formulae hold for the stress corresponding to the waves running to the right and left in the region $s < x < s+d$

$$\begin{aligned} \sigma_{AS}(x,t) &= -E_S\frac{i\omega}{c_S}A_S^*\exp i\omega\left(t - \frac{x-s}{c_S}\right), \\ \sigma_{BS}(x,t) &= +E_S\frac{i\omega}{c_S}B_S^*\exp i\omega\left(t + \frac{x-s}{c_S}\right). \end{aligned} \tag{39}$$

There follows

$$L_{A0} = \frac{\omega^2 E_0 A_0\overline{A_0}}{2c_0^2}, \quad L_{B0} = \frac{\omega^2 E_0 B_0\overline{B_0}}{2c_0^2 E_0}, \tag{40}$$

$$L_{AS} = \frac{\omega^2 E_S A_S^*\overline{A_S^*}}{2c_S^2}, \quad L_{BS} = \frac{\omega^2 E_S B_S^*\overline{B_S^*}}{2c_S^2}. \tag{41}$$

Calculate in turn the elastic energy of the layers. The wave of amplitude A_0 carries some energy. Denote by Q_{A0} the energy of the layer thickness equal to the distance c_0 travelled by this wave in the region $x < 0$ in the unit time. The energy Q_{B0} is the energy of the thickness equal to the distance c_0 travelled by the wave of amplitude B_0 in the same region in unit time. Similarly calculate the elastic energies Q_{AS}, Q_{BS} of a length equal to the distance c_S travelled by the wave in the unit time in the region $x > s$. Since the distance travelled by the vaves in unit time in region $x < 0$ and $s < x < s+d$ equals respectively c_0 and c_S, the energy densities (40) and (41) must be multiplied by c_0 and c_S, respectively. Therefore

$$Q_{A0} = \frac{\omega^2 E_0 A_0\overline{A_0}}{2c_0}, \quad Q_{B0} = \frac{\omega^2 E_0 B_0\overline{B_0}}{2c_0}, \tag{42}$$

$$Q_{AS} = \frac{\omega^2 E_S A_S^*\overline{A_S^*}}{2c_S}, \quad Q_{BS} = \frac{\omega^2 E_S B_S^*\overline{B_S^*}}{2c_S}. \tag{43}$$

Comparison with (36) proves that

$$Q_{A0} - Q_{B0} = Q_{AS} - Q_{BS}. \tag{44}$$

Therefore the relation (29) expresses the conservation of elastic energy of the waves. The total energy $Q_{A0} - Q_{B0}$ travelling to the right in the region $x < 0$ equals the total energy $Q_{AS} - Q_{BS}$ travelling to the right in the region $s < x > s+d$.

5. Infinitesimally thin extra layer

We base on the expressions derived in Section 2 for the amplitudes A_S^*, B_S^* in the virtual layer. According to the calculations of Section 2 these amplitudes do not depend on the layer thicknesss d. Analyse the right inhomogeneous region $s+d < x < h+d$. The continuity of displacement and stress at $x = s+d$, cf. Fig 2 lead to the relations

$$\begin{aligned} &A_S^*\exp i\omega\left(-\frac{d}{c_S}\right) + B_S^*\exp i\omega\left(\frac{d}{c_S}\right) = \\ &\qquad A_R^*\alpha(s) + B_R^*\beta(s), \\ &-\frac{i\omega}{c_S}A_S^*\exp i\omega\left(-\frac{d}{c_S}\right) + \frac{i\omega}{c_S}B_S^*\exp i\omega\left(\frac{d}{c_S}\right) = \\ &\qquad A_R^*\alpha'(s) + B_R^*\beta'(s). \end{aligned} \tag{45}$$

The amplitudes A_S^*, B_S^* are known functions of the amplitudes A_0, B_0, which are given in advance. The above relations allow to calculate the amplitudes A_R^*, B_R^* for each d. It must be noted that the extra layer was artificially introduced with the only purpose to discover the meaning to the displacement in the inhomogeneous region. The extra layer will fulfill its duty for each d. Especially simple formulae are obtained if the thickness d is infinitesimal. Therefore instead of arbitrary d we assume $d \to 0$. In such situation relations (45) reduce to

$$\begin{aligned} A_S^* + B_S^* &= A_R^*\alpha(s) + B_R^*\beta(s) \ , \\ -\frac{i\omega}{c_S}A_S^* + \frac{i\omega}{c_S}B_S^* &= A_R^*\alpha'(s) + B_R^*\beta'(s) \ . \end{aligned} \tag{46}$$

Note that (46) holds not only for $d \to 0$, but also for the more general situation

$$\omega\frac{d}{c_S} = 2\pi. \tag{47}$$

corresponding to the finite thickness of the extra layer.

The left-hand side of the system of Eqs. (46) coincides with the right-hand side of the system (22). Therefore,

$$\begin{aligned} (A_R^* - A_L^*)\,\alpha(s) + (B_R^* - B_L^*)\,\beta(s) \ &= 0 \ , \\ (A_R^* - A_L^*)\,\alpha'(s) + (B_R^* - B_L^*)\,\beta'(s) &= 0. \end{aligned} \tag{48}$$

The values $\alpha(s)$, $\beta(s)$, $\alpha'(s)$, $\beta'(s)$ are known, and the amplitudes A_R^*, B_R^* must be calculated. Since $\alpha(\xi)$, $\beta(s)$ are linearly independent there is

$$A_R^* = A_L^*, \ \ B_R^* = B_L^*. \tag{49}$$

The amplitudes in the left and the right regions are the same. The presence of the infinitesimal extra layer does not influence the time-dependent displacement. When calculating the displacement in the homogeneous region $x > h$ we face the same situation as in Chapter 2. The formulae (23) and (24) may be used again, but instead of $x = s$ must be taken $x = h$. The following expressions for A_2^*, B_2^* are obtained

$$\begin{aligned} 2M_0A_2^* = &\Big\{-[\alpha_0'\beta_h - \beta_0'\alpha_h] + \frac{c_h}{i\omega}[\alpha_0'\beta_h' - \beta_0'\alpha_h'] \\ &- \frac{i\omega}{c_0}[\alpha_0\beta_h - \beta_0\alpha_h] + \frac{c_h}{c_0}[\alpha_0\beta_h' - \beta_0\alpha_h']\Big\}A_0^* \\ &+\Big\{-[\alpha_0'\beta_h - \beta_0'\alpha_h] + \frac{c_h}{i\omega}[\alpha_0'\beta_h' - \beta_0'\alpha_h'] \\ &+\frac{i\omega}{c_0}[\alpha_0\beta_h - \beta_0\alpha_h] - \frac{c_h}{c_0}[\alpha_0\beta_h' - \beta_0\alpha_h']\Big\}B_0^*, \\ 2M_0B_2^* = &\Big\{-[\alpha_0'\beta_h - \beta_0'\alpha_h] - \frac{c_h}{i\omega}[\alpha_0'\beta_h' - \beta_0'\alpha_h'] \\ &-\frac{i\omega}{c_0}[\alpha_0\beta_h - \beta_0\alpha_h] - \frac{c_h}{c_0}[\alpha_0\beta_h' - \beta_0\alpha_h']\Big\}A_0^* \\ &+\Big\{-[\alpha_0'\beta_h - \beta_0'\alpha_h] - \frac{c_h}{i\omega}[\alpha_0'\beta_h' - \beta_0'\alpha_h'] \\ &+\frac{i\omega}{c_0}[\alpha_0\beta_h - \beta_0\alpha_h] + \frac{c_h}{c_0}[\alpha_0\beta_h' - \beta_0\alpha_h']\Big\}B_0^*. \end{aligned} \tag{50}$$

The complex-valued A_2^*, B_2^* are the amplitudes of the two waves running to the right and to the left, respectively, in the homogeneous region $x > h$.

Pass to the calculation of the two products $A_h^*\overline{A_h^*}$, $B_h^*\overline{B_h^*}$. The first is equal to the squared real amplitude of the wave running to the right. The second is equal to the squared real amplitude of the wave running to the left. In accord with (23) and (24) we have

$$\begin{aligned} &4M_0^2A_2^*\overline{A_2^*} = \\ &\Bigg\{\Big[(\alpha_0'\beta_h - \beta_0'\alpha_h) - \frac{c_h}{c_0}(\alpha_0\beta_h' - \beta_0\alpha_h')\Big]^2 \\ &+\Big[\frac{\omega}{c_0}(\alpha_0\beta_h - \beta_0\alpha_h) + \frac{c_h}{\omega}(\alpha_0'\beta_h' - \beta_0'\alpha_h')\Big]^2\Bigg\}A_0^*\overline{A_0^*} \\ &+\Bigg\{\Big[(\alpha_0'\beta_h - \beta_0'\alpha_h) + i\frac{\omega}{c_0}(\alpha_0\beta_h - \beta_0\alpha_h)\Big]^2 \\ &+\Big[\frac{c_h}{\omega}(\alpha_0'\beta_h' - \beta_0'\alpha_h') + i\frac{c_h}{c_0}(\alpha_0\beta_h' - \beta_0\alpha_h')\Big]^2\Bigg\}A_0^*\overline{B_0^*} \\ &+\Bigg\{\Big[(\alpha_0'\beta_h - \beta_0'\alpha_h) - i\frac{\omega}{c_0}(\alpha_0\beta_h - \beta_0\alpha_h)\Big]^2 \\ &+\Big[\frac{c_h}{\omega}(\alpha_0'\beta_h' - \beta_0'\alpha_h') - i\frac{c_h}{c_0}(\alpha_0\beta_h' - \beta_0\alpha_h')\Big]^2\Bigg\}\overline{A_0^*}B_0^* \\ &+\Bigg\{\Big[(\alpha_0'\beta_h - \beta_0'\alpha_h) + \frac{c_h}{c_0}(\alpha_0\beta_h' - \beta_0\alpha_h')\Big]^2 \\ &+\Big[\frac{\omega}{c_0}(\alpha_0\beta_h - \beta_0\alpha_h) - \frac{c_h}{\omega}(\alpha_0'\beta_h' - \beta_0'\alpha_h')\Big]^2\Bigg\}B_0^*\overline{B_0^*}, \\ &4M_0^2B_h^*\overline{B_h^*} = \\ &\Bigg\{\Big[(\alpha_0'\beta_h - \beta_0'\alpha_h) - \frac{c_h}{c_0}(\alpha_0\beta_h' - \beta_0\alpha_h')\Big]^2 \\ &+\Big[\frac{\omega}{c_0}(\alpha_0\beta_h - \beta_0\alpha_h) + \frac{c_h}{\omega}(\alpha_0'\beta_h' - \beta_0'\alpha_h')\Big]^2\Bigg\}A_0^*\overline{A_0^*} \\ &+\Bigg\{\Big[(\alpha_0'\beta_h - \beta_0'\alpha_h) + i\frac{\omega}{c_0}(\alpha_0\beta_h - \beta_0\alpha_h)\Big]^2 \\ &+\Big[\frac{c_h}{\omega}(\alpha_0'\beta_h' - \beta_0'\alpha_h') + i\frac{c_h}{c_0}(\alpha_0\beta_h' - \beta_0\alpha_h')\Big]^2\Bigg\}A_0^*\overline{B_0^*} \\ &+\Bigg\{\Big[(\alpha_0'\beta_h - \beta_0'\alpha_h) - i\frac{\omega}{c_0}(\alpha_0\beta_h - \beta_0\alpha_h)\Big]^2 \\ &+\Big[\frac{c_h}{\omega}(\alpha_0'\beta_h' - \beta_0'\alpha_h') - i\frac{c_h}{c_0}(\alpha_0\beta_h' - \beta_0\alpha_h')\Big]^2\Bigg\}\overline{A_0^*}B_0^* \\ &+\Bigg\{\Big[(\alpha_0'\beta_h - \beta_0'\alpha_h) + \frac{c_h}{c_0}(\alpha_0\beta_h' - \beta_0\alpha_h')\Big]^2 \\ &+\Big[\frac{\omega}{c_0}(\alpha_0\beta_h - \beta_0\alpha_h) - \frac{c_h}{\omega}(\alpha_0'\beta_h' - \beta_0'\alpha_h')\Big]^2\Bigg\}B_0^*\overline{B_0^*}. \end{aligned} \tag{51}$$

The above products express the power of the transmitted waves.

REFERENCES

[1] Z. Wesołowski, "Wave reflection on a continuous transition zone between two homogeneous materials", *Acta Mechanica* 105, 119–131 (1994).

[2] S. Kaliski, *Vibrations and Waves*, vol III, PWN, Warszawa, 1986, (in Polish).

[3] E. Kamke, *Differentialgleichungen*, Akademische Verlagsgesellschaft, Leipzig, 1943.