BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 55, No. 4, 2007

Positive 2D hybrid linear systems

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Abstract. A new class of positive hybrid linear systems is introduced. The solution of the hybrid system is derived and necessary and sufficient condition for the positvity of the class of hybrid systems are established. The classical Cayley-Hamilton theorem is extended for the hybrid systems. The reachability of the hybrid system is considered and sufficient conditions for the reachability are established. The considerations are illustrated by a numerical example.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [1,2]. Recent developments in positive systems theory and some new results are given in [3]. The realization problem for positive discrete-time and continuous-time systems without and with delays was considered in [1,2,4,5–9].

The reachability, controllability and minimum energy control of positive linear discrete-time systems with delays have been considered in [10].

The relative controllability of stationary hybrid systems has been investigated in [11] and the observability of linear differential-algebraic systems with delays has been considered in [12].

The main purpose of this paper is to introduce a class of positive 2D hybrid systems. A solution to the hybrid system will be derived and necessary and sufficient condition for the positivity will be established. The classical Cayley-Hamilton theorem will be extended for hybrid systems and sufficient conditions for the reachability will be established.

To the best knowledge of the author the positive hybrid systems has not been considered yet.

2. Equations of the hybrid systems and their solutions

Let $R^{n\times m}$ be the set of $n \times m$ matrices with entries form the field of real number R and Z_+ be the set of non-negative integers. The $n \times n$ identity matrix will be denoted by I_n.

Consider a hybrid system described by the equations

$$\dot{x}_1(t,i) = A_{11}x_1(t,i) + A_{12}x_2(t,i) + B_1u(t,i), \quad t \in R_+ = [0, +\infty] \tag{1a}$$

$$x_2(t,i+1) = A_{21}x_1(t,i) + A_{22}x_2(t,i) + B_2u(t,i), \quad i \in Z_+ \tag{1b}$$

$$y(t,i) = C_1x_1(t,i) + C_2x_2(t,i) + Du(t,i) \tag{1c}$$

where $\dot{x}_1(t,i) = \frac{\partial x_1(t,i)}{\partial t}$, $x_1(t,i) \in R^{n_1}$, $x_2(t,i) \in R^{n_2}$, $u(t,i) \in R^m$, $y(t,i) \in R^p$ and $A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2, D$ are real matrices with appropriate dimensions.
Boundary conditions for (1a) and (1b) have the form

$$x_1(0,i) = x_1(i),\ i \in Z_+ \text{ and } x_2(t,0) = x_2(t),\ t \in R_+ \tag{2}$$

Note that the hybrid system (1) has a similar structure as the Roesser model [2,13,14].

Theorem 1. Solutions to the Eqs. (1a) and (1b) with given boundary conditions (2) have the forms

$$x_1(t,i) = \begin{cases} \Phi(t)x_1(0) + P_tx_2(t) + Q_tu(t,0) \quad for \quad i=0 \\ \Phi(t)x_1(i) + \sum_{k=0}^{i-1} P_t(A_{21}P_t + A_{22})^{i-k-1} \\ [A_{21}\Phi(t)x_1(k) + (A_{21}Q_t + B_2)u(t,k)] \\ +P_t(A_{21}P_t + A_{22})^i x_2(t) + Q_tu(t,i) \quad for \quad i = 1,2,... \end{cases} \tag{3a}$$

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$$\begin{aligned} x_2(t,i) &= \sum_{k=0}^{i-1}(A_{21}P_t+A_{22})^{i-k-1} \\ &[A_{21}\Phi(t)x_1(k)+(A_{21}Q_t+B_2)u(t,k)] \\ &+(A_{21}P_t+A_{22})^i x_2(t) \quad for \quad i=1,2,... \end{aligned} \tag{3b}$$

where $\Phi(t)=e^{A_{11}t}$ and the operators P_t and Q_t are defined by:

$$\begin{aligned} P_t x &= \int_0^t \Phi(t-\tau)A_{12}x(\tau)d\tau, \\ Q_t x &= \int_0^t \Phi(t-\tau)B_1 x(\tau)d\tau \end{aligned} \tag{4}$$

Proof. The proof will be accomplished by induction with respect to i.
From (1a) we have

$$x_1(t,i)=\Phi(t)x_1(i)+P_t x_2(t,i)+Q_t u(t,i) \tag{5}$$

From (1b) for $i=0$ we have

$$x_2(t,1)=A_{21}x_1(t,0)+A_{22}x_2(t)+B_2u(t,0) \tag{6}$$

and from (5) for $i=0$

$$x_1(t,0)=\Phi(t)x_1(0)+P_t x_2(t)+Q_t u(t,0) \tag{7}$$

Substitution of (7) into (6) yields

$$\begin{aligned} x_2(t,1) &= A_{21}\Phi(t)x_1(0)+ \\ &+(A_{21}P_t+A_{22})x_2(t)+(A_{21}Q_t+B_2)u(t,0) \end{aligned} \tag{8}$$

The same result we obtain from (3b) for $i=1$.
Likewise substituting (8) into the equation (obtained from (5) for $i=1$)

$$x_1(t,1)=\Phi(t)x_1(1)+P_t x_2(t,1)+Q_t u(t,1)$$

we obtain

$$\begin{aligned} x_1(t,1) &= P_t A_{21}\Phi(t)x_1(0)+\Phi(t)x_1(1)+ \\ &+P_t(A_{21}P_t+A_{22})x_2(t)+P_t(A_{21}Q_t+B_2)u(t,0)+ \\ &+Q_t u(t,1) \end{aligned}$$

The same result we obtain from (3a) for $i=1$. Therefore, the hypothesis is true for $i=1$. Assuming that the hypothesis is true for $i=k$ we shall show that it is also true for $i=k+1$.

Using (3a) and (3b) for $i=k>1$ we may write

$$\begin{aligned} &A_{21}x_1(t,k)+A_{22}x_2(t,k)+B_2u(t,k) \\ &= A_{21}\{\Phi(t)x_1(k)+\sum_{j=0}^{k-1}P_t(A_{21}P_t+A_{22})^{k-j-1} \\ &[A_{21}\Phi(t)x_1(j)+(A_{21}Q_t+B_2)u(t,j)] \\ &+P_t(A_{21}P_t+A_{22})^k x_2(t)+Q_t u(t,k)\} \\ &+A_{22}\{\sum_{j=0}^{k-1}(A_{21}P_t+A_{22})^{k-j-1}[A_{21}\Phi(t)x_1(j) \\ &+(A_{21}Q_t+B_2)u(t,j)+(A_{21}P_t+A_{22})^k x_2(t)]\}+B_2u(t,k) \\ &= \sum_{j=0}^{k}(A_{21}P_t+A_{22})^{k-j} \\ &[A_{21}\Phi(t)x_1(j)+(A_{21}Q_t+B_2)u(t,j)] \\ &+(A_{21}P_t+A_{22})^{k+1}x_2(t)=x_2(t,k+1) \end{aligned}$$

Likewise using (5), (3b) and (3a) for $i=k>1$ we obtain

$$\begin{aligned} &\Phi(t)x_1(k+1)+P_t x_2(t,k+1)+Q_t u(t,k+1)= \\ &\Phi(t)x_1(k+1)+P_t\{\sum_{j=0}^{k}(A_{21}P_t+A_{22})^{k-j} \\ &[A_{21}\Phi(t)x_1(j)+(A_{21}Q_t+B_2)u(t,j)] \\ &+(A_{21}P_t+A_{22})^{k+1}x_2(t)\}+Q_t u(t,k+1) \\ &= \Phi(t)x_1(k+1)+\sum_{j=0}^{k}P_t(A_{21}P_t+A_{22})^{k-j}[A_{21}\Phi(t)x_1(j) \\ &+(A_{21}Q_t+B_2)u(t,j)]+P_t(A_{21}P_t+A_{22})^{k+1}x_2(t) \\ &+Q_t u(t,k+1)=x_1(t,k+1) \end{aligned}$$

This completes the proof. □

3. Positive hybrid systems

Let $R_+^{n\times m}$ be the set of $n\times m$ real matrices with nonnegative entries and $R_+^n=R_+^{n\times 1}$.

Definition 1. The hybrid system (1) is called internally positive if $x_1(t,i)\in R_+^{n_1}$, $x_2(t,i)\in R_+^{n_2}$, and $y(t,i)\in R_+^p$, $t\in R_+$, $i\in Z_+$ for arbitrary boundary conditions $x_1(i)\in R_+^{n_1}$, $i\in Z_+$, $x_2(t)\in R_+^{n_2}$, $t\in R_+$ and inputs $u(t,i)\in R_+^m$, $t\in R_+$, $i\in Z_+$.

Let M_n be the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries).

Theorem 2. The hybrid system (1) is internally positive if and only if

$$\begin{aligned} &A_{11}\in M_{n_1}, A_{12}\in R_+^{n_1\times n_2}, A_{21}\in R_+^{n_2\times n_1}, A_{22}\in R_+^{n_2\times n_2}, \\ &B_1\in R_+^{n_1\times m}, B_2\in R_+^{n_2\times m}, \\ &C_1\in R_+^{p\times n_1}, C_2\in R_+^{p\times n_2}, D\in R_+^{p\times m} \end{aligned} \tag{9}$$

Proof. Sufficiency. It is well-known [1] that $\Phi(t) = e^{A_{11}t} \in R_+^{n_1 \times n_1}$ if and only if A_{11} is the Metzler matrix. Thus, from (3), (4) and (1c) it follows that if the condition (9) is satisfied, then $x_1(t,i) \in R_+^{n_1}$, $x_2(t,i) \in R_+^{n_2}$, and $y(t,i) \in R_+^p$ for all $x_1(i) \in R_+^{n_1}$, $x_2(t) \in R_+^{n_2}$ and $u(t,i) \in R_+^m$, $t \in R_+$, $i \in Z_+$.
Necessity. Let $u(t,0) = 0$, $x_2(t) = 0$, $t \in R_+$ and $x_1(0) = e_i$ (the ith column of the identity matrix I_{n_1}). From (1a) for $i = 0$, $t \in R_+$ and (7) we have $\dot{x}_1(t,0) = A_{11}\Phi(t)e_i \geq 0$ and the trajectory does not leave the orthant $R_+^{n_1}$ only if $\dot{x}_1(0,0) = A_{11}e_1 \in R_+^{n_1}$, what implies $a_{ij} \geq 0$ for $i \neq j$ and A_{11} has to be a Metzler matrix, i.e. $A_{11} \in M_{n_1}$. For the same reasons for $x_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_1(0,0) = B_1 u(0,0) \in R_+^{n_1}$, what implies $B_1 \in R_+^{n_1 xm}$ since $u(0,0) \in R_+^m$ may be arbitrary. Similarly, for $x_1(0) = 0$, $u(0,0) = 0$, $\dot{x}_1(0,0) = A_{12}x_2(0) \in R_+^{n_1}$ what implies $A_{12} \in R_+^{n_1 x n_2}$ since $x_2(0,0)$ may be arbitrary. From (1c) for $x_2(0) = 0$, $u(0,0) = 0$ we have $y(0,0) = C_1 x_1(0) \in R_+^p$, what implies $C_1 \in R_+^{pxn_1}$, since $x_1(0) \in R_+^{n_1}$ may be arbitrary.
The proof for A_{21}, A_{22}, B_2, C_2 and D is similar. □

Definition 2. The hybrid system (1) is called externally positive if $y(t,i) \in R_+^p$ for all inputs $u(t,i) \in R_+^m$, $t \in R_+$, $i \in Z_+$ and zero boundary conditions (2).

From comparison of Definition 1 and 2 it follows that every internally positive hybrid system (1) is also externally positive.

The output of the system with zero boundary conditions (2) for the input $u(t,i) = \delta(t)$, where $\delta(t)$ is the Dirac impulse is called the impulse response $g(t,i)$ of a single-input single-output hybrid system (1). Assuming that only one input is equal to $\delta(t)$ and the remaining inputs are zero we may define the matrix of impulse responses $g(t,i) \in R^{pxm}$ of the hybrid system (1).

Theorem 3. The hybrid system (1) is externally positive if and only if its matrix of impulse response is nonnegative

$$g(t,i) \in R_+^{pxn} \text{ for } t \in R_+,\ i \in Z_+. \tag{10}$$

Proof. The necessity of the condition (10) immediately follows from Definition 2. From (1c) and (3) for $x_1(i) = 0$, $x_2(t) = 0$ and $u(t,i) = \delta(t)$, $t \in R_+$, $i \in Z_+$ we have

$$\begin{aligned} &g(t,i) = \\ &\quad (C_1 P_t + C_2)(A_{21}P_t + A_{22})^{i-1}(A_{21}\Phi(t)B_1 + B_2\delta(t)) \\ &\quad + \Phi(t)B_1 + D\delta(t),\ t \in R_+,\ i \in Z_+ \end{aligned} \tag{11}$$

since $Q_t\delta(t) = \int_0^t \Phi(t-\tau)B_1\delta(\tau)d\tau = \Phi(t)B_1$

and $P_t = \begin{cases} (e^{A_{11}t} - I_{n_1})A_{11}^{-1}A_{12} & if \quad \det A_{11} \neq 0 \\ \int_0^t e^{A_{11}\tau}A_{12}d\tau & if \quad \det A_{11} = 0 \end{cases}$

From (1c), (3) and (11) it follows

$$\begin{aligned} y(t,i) &= C_1 x_1(t,i) + C_2 x_2(t,i) + Du(t,i) = \\ &= \sum_{k=0}^{i-1} g(t,i-k)u(t,k) \end{aligned} \tag{12}$$

If the condition (10) is satisfied, then from (12) we have $y(t,i) \in R_+^p$ for all $u(t,i) \in R_+^m$, $t \in R_+$, $i \in Z_+$. This completes the proof. □

4. Extension of the Cayley-Hamilton theorem for hybrid systems

The equations (1a) and (1b) for $B_1 = 0$ and $B_2 = 0$ take the form

$$\begin{aligned} \dot{x}_1(t,i) &= A_{11}x_1(t,i) + A_{12}x_2(t,i) \\ x_2(t,i+1) &= A_{21}x_1(t,i) + A_{22}x_2(t,i) \end{aligned} \quad t \in R_+, i \in Z_+ \tag{13}$$

Using the Laplace transform with respect to t and Z transform will respect to i for (13) and eliminating from the equations $x_2(s,z)$, we obtain

$$[I_{n_1}s - A_{11} - A_{12}[I_{n_2}z - A_{22}]^{-1}A_{21}]x_1(s,z) = 0 \tag{14}$$

Let $d(z) = \det[I_{n_2}z - A_{22}]$ be the characteristic polynomial of the matrix A_{22} and $B(z) = Adj[I_{n_2}z - A_{22}]$ be the adjoint matrix. Then we have

$$\begin{aligned} &[I_{n_1}s - A_{11} - A_{12}[I_{n_2}z - A_{22}]^{-1}A_{21}]^{-1} \\ &= [d^{-1}(z)[(I_{n_1}s - A_{11})d(z) - A_{12}B(z)A_{21}]]^{-1} \\ &= d(z)[(I_{n_1}s - A_{11})d(z) - A_{12}B(z)A_{21}]^{-1} \end{aligned} \tag{15}$$

and

$$H(s,z) = d(s,z)\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Phi_{ij}s^{-(i+1)}z^{-(j+1)}\right) \tag{16}$$

$$\begin{aligned} H(s,z) &= Adj[(I_{n_1}s - A_{11})d(z) - A_{12}B(z)A_{21}] \\ &= \sum_{k=0}^{N-1}\sum_{l=0}^{M-1} H_{kl}s^k z^l \end{aligned} \tag{17}$$

$$\begin{aligned} d(s,z) &= \det[(I_{n_1}s - A_{11})d(z) - A_{12}B(z)A_{21}] \\ &= \sum_{k=0}^{N}\sum_{l=0}^{M} a_{kl}s^k z^l, \quad (a_{NM} = 1) \end{aligned} \tag{18}$$

$$\begin{aligned} &[(I_{n_1}s - A_{11})d(z) - A_{12}B(z)A_{21}]^{-1} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Phi_{ij}s^{-(i+1)}z^{-(j+1)} \end{aligned} \tag{19}$$

Knowing the coefficient matrices H_{kl}, $k = 0,1,...,N-1$; $l = 0,1,...,M-1$ of (17) and the coefficients a_{kl}, $k = 0,1,...,N$; $l = 0,1,...,M$ of (18), we may find the matrices Φ_{ij}, for $i = 0,1,...,M$; $j = 0,1,...,N$, as follows.

Comparison of coefficients at the same powers of s and z of the equality (16) yields $H_{N-1,M-1} = \Phi_{00}$, $H_{N-1,M-2} = \Phi_{01} + a_{N,M-1}\Phi_{00}$, $H_{N-2,M-1} = \Phi_{10} + a_{N-1,M}\Phi_{00}\ldots$

$$\begin{aligned} \Phi_{00} &= H_{N-1,M-1}, \\ \Phi_{01} &= H_{N-1,M-2} - a_{N,M-1}\Phi_{00}, \\ \Phi_{10} &= H_{N-2,M-1} - a_{N-1,M}\Phi_{00}\ldots \end{aligned} \tag{20}$$

Theorem 4. The matrices Φ_{ij}, defined by (19) satisfy the following equations

$$\sum_{k=0}^{N}\sum_{l=0}^{M} a_{kl}\Phi_{k+v,l+w} = 0 \quad \text{for} \quad v, w = -1, 0, 1, \ldots \ (v+w \neq -2) \tag{21}$$

where a_{kl} are the coefficient of the polynomial (18).

Proof. Note that the adjoint matrix (17) is a polynomial matrix in s and z with nonnegative powers. Comparison of the matrix coefficients at the powers $s^{-(v+1)}$ and $z^{-(w+1)}$ of the equality (16) yields (21).

The equation (21) is an extension of the classical Cayley-Hamilton theorem for the hybrid system (13). In particular case from (21) for $v = w = 0$, we obtain

$$\sum_{k=0}^{N}\sum_{l=0}^{M} a_{kl}\Phi_{k,l} = 0 \tag{22}$$

Example 1. Consider the system (13) with the matrices

$$A_{11} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}, A_{22} = [1] \tag{23}$$

In this case we have

$$\begin{aligned} d(s,z) &= \det[(I_{n_1}s - A_{11})d(z) - A_{12}B(z)A_{21}] \\ &= \det\begin{bmatrix} (s+1)(z-1) & 0 \\ 0 & (s+2)(z-1)-1 \end{bmatrix} \\ &= (sz - s + z - 1)(sz - s + 2z - 3) \\ &= s^2z^2 - 2s^2z + 3sz^2 + s^2 + 2z^2 - 7sz + 4s - 5z + 3 \end{aligned} \tag{24}$$

$$\begin{aligned} H(s,z) &= Adj[(I_{n_1}s - A_{11})d(z) - A_{12}B(z)A_{21}] \\ &= \begin{bmatrix} sz - s + 2z - 3 & 0 \\ 0 & sz - s + z - 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} sz + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} s + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} z \\ &+ \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{25}$$

Using (20) we obtain

$$\Phi_{00} = H_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Phi_{10} = H_{01} - a_{12}\Phi_{00} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},$$

$$\Phi_{01} = H_{10} - a_{21}\Phi_{00} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{aligned} \Phi_{11} &= H_{00} - a_{12}\Phi_{01} - a_{21}\Phi_{10} - a_{11}\Phi_{00} \\ &= \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \\ &+7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \end{aligned}$$

$$\Phi_{21} = -a_{12}\Phi_{11} - \Phi_{01} = -3\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$\Phi_{12} = -a_{21}\Phi_{11} - \Phi_{10} = 2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Phi_{20} = -a_{12}\Phi_{10} - a_{02}\Phi_{00} = -3\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix},$$

$$\Phi_{02} = -a_{21}\Phi_{01} - a_{20}\Phi_{00} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{aligned} \Phi_{22} &= -a_{12}\Phi_{12} - a_{21}\Phi_{21} - a_{20}\Phi_{20} - a_{02}\Phi_{02} - a_{10}\Phi_{10} \\ &-a_{11}\Phi_{11} - a_{01}\Phi_{01} - a_{00}\Phi_{00} \\ &= -3\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &+7\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - 4\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &-3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

From (22) we have

$$\sum_{k=0}^{2}\sum_{l=0}^{2} a_{kl}\Phi_{k,l} = 3\Phi_{00} - 5\Phi_{01} + 4\Phi_{10} - 7\Phi_{11}$$
$$+2\Phi_{02} + \Phi_{20} + 3\Phi_{12} - 2\Phi_{21} + \Phi_{22}$$
$$= 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$
$$-7\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$
$$+3\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

5. Reachability of the hybrid systems

Consider the hybrid system (1) with zero boundary conditions (ZBC) (2).

Definition 3. A state

$$x_f = \begin{bmatrix} x_{1f} \\ x_{2f} \end{bmatrix} \in R_+^{n_1+n_2} \tag{26}$$

of the positive hybrid system (1) with ZBC is called reachable in time t_f if there exists an input $u(t,i) \in R_+^m$ for $t \in [0, t_f]$, $i \in Z_+$ such that

$$\begin{bmatrix} x_1(t_f, 0) \\ x_2(t_f, 2) \end{bmatrix} = \begin{bmatrix} x_{1f} \\ x_{2f} \end{bmatrix}. \tag{27}$$

If every state $x_f \in R_+^{n_1+n_2}$ is reachable then the system (1) is called reachable.

A matrix is called monomial if its every row and its every column contain only one positive entry and the remaining entries are zero.

Let us assume that

A1) the matrix

$$R_f = \int_0^t \Phi(\tau) B_1 B_1^T \Phi^T(\tau) d\tau \tag{28}$$

is monomial,

A2) the vector

$$\hat{x}_{2f} = x_{2f} - A_{21}\int_0^{t_f} \Phi(t_f - \tau) A_{12}$$
$$[A_{21}\int_0^{\tau} \Phi(\tau - \tau_1) B_1 u(\tau_1, 0) d\tau_1 + B_2 u(\tau, 0) d\tau] \tag{29}$$
$$-A_{22}[A_{21}x_{1f} + B_2 u(t_f, 0)]$$

has nonnegative components, $\hat{x}_{2f} \in R_+^{n_2}$, where

$$u(t,0) = B_1^T \Phi(t_f - t) R_f^{-1} x_{1f}, \ \ t \in [0, t_f]. \tag{30}$$

Theorem 5. The state (26) satisfying the condition $\hat{x}_{2f} \in R_+^{n_2}$ of the positive hybrid system (1) with ZBC is reachable in time t_f if the assumption A1) is met,

$$rank[P(t_f) + B_2, \hat{x}_{2f}] = rank[P(t_f) + B_2] \tag{31}$$

and the equation

$$\hat{x}_{2f} = (P(t_f) + B_2)u_1, \ \ P(t_f) = A_{21}\int_0^{t_f} \Phi(t) B_1 dt \tag{32}$$

has a nonnegative solution $u_1 \in R_+^m$.

Remark 1. If the matrix $P(t_f) + B_2$ is square then it should be monomial.

Proof. If the matrix (28) is monomial then $R_f^{-1} \in R_+^{n_1 \times n_1}$ and the input (30) steers the state of the subsystem (1a) from ZBC to the desired final state x_{1f}. Using (5) for $i = 0$ and (30), (28) and $\Phi(t) = e^{A_{11}t}$ we obtain

$$x_1(t_f, 0) = \int_0^{t_f} \Phi(t_f - \tau) B_1 u(\tau, 0) d\tau$$
$$= \int_0^{t_f} e^{A_{11}(t_f - \tau)} B_1 B_1^T e^{A_{11}^T (t_f - \tau)} d\tau R_f^{-1} x_{1f}$$
$$= \int_0^{t_f} e^{A_{11}\tau} B_1 B_1^T e^{A_{11}^T \tau} d\tau R_f^{-1} x_{1f} = x_{1f}$$

From (1b) for $i = 0$ and $t = t_f$, we have

$$x_2(t_f, 1) = A_{21}x_{1f} + B_2 u(t_f, 0) \tag{33}$$

since $x_2(t_f, 0) = x_2(t_f) = 0$ and $x_1(t_f, 0) = x_{1f}$. Using (30) for $t = t_f$ and (33) we may find $x_2(t_f, 1)$.

From (5) for $i = 1$, $t = t_f$ and $x_1(1) = 0$ and using (4) we obtain

$$x_1(t_f, 1) = \int_0^{t_f} \Phi(t_f - \tau) A_{12} x_2(\tau, 1) d\tau + \int_0^{t_f} \Phi(t_f - \tau) B_1 u(\tau, 1) d\tau$$
$$= \int_0^{t_f} \Phi(t_f - \tau) A_{12} [A_{21} x_1(\tau, 0) + B_2 u(\tau, 0)] d\tau + \int_0^{t_f} \Phi(t_f - \tau) B_1 u(\tau, 1) d\tau \quad (34)$$
$$= \bar{x}_1(t_f, 1) + \int_0^{t_f} \Phi(t_f - \tau) B_1 u(\tau, 1) d\tau$$

where

$$\bar{x}_1(t_f, 1) = \int_0^{t_f} \Phi(t_f - \tau) A_{12} [A_{21} x_1(\tau, 0) + B_2 u(\tau, 0)] d\tau = \int_0^{t_f} \Phi(t_f - \tau) A_{12} [A_{21} \int_0^{\tau} \Phi(\tau - \tau_1) B_1 u(\tau_1, 0) d\tau_1 + B_2 u(\tau, 0)] d\tau \quad (35)$$

is known for given (30).
Let

$$u(t, 1) = u_1 \ \text{ for } \ t \in [0, t_f]. \quad (36)$$

Then from (1b) for $i = 1$, $t = t_f$ and (34) we obtain

$$x_2(t_f, 2) = A_{21} x_1(t_f, 1) + A_{22} x_2(t_f, 1) + B_2 u(t_f, 1) = A_{21} \bar{x}_1(t_f, 1) + A_{22} x_2(t_f, 1) + (A_{21} \int_0^{t_f} \Phi(\tau) d\tau B_1 + B_2) u_1$$

and the Eq. (32).

If the assumptions A1) and the condition (31) are met and Eq. (32) has a nonnegative solution, then $u_1 \in R_+^m$.

This completes the proof.

Remark 2. Let the matrix

$$W_f = \int_0^{t_f} \Phi(\tau) B_1 d\tau \quad (37)$$

be monomial and $u(t, 0) = u_0$ for $t \in [0, t_f]$.

Then the input vector $u_0 \in R_+^m$ can be computed from the formula

$$u_0 = W_f^{-1} x_{1f} \quad (38)$$

which follows from the equality

$$x_{1f} = \int_0^{t_f} \Phi(\tau) B_1 u(\tau, 0) d\tau = W_f u_0. \quad (39)$$

Remark 3. Note that the reachability depends only on the matrices A_{kl}, B_k, $k, l = 1, 2$ and it is independent of the remaining matrices of the system (1).

Example 2. Consider the hybrid system (1) with

$$A_{11} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, A_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, A_{21} = \begin{bmatrix} 1 & 2 \end{bmatrix}, A_{22} = [2], B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}. \quad (40)$$

Compute the input $u(t, i)$ for $t \in [0, t_f]$, $i = 0, 1$, $t_f = 1$ which steers the state of the system from ZBC to the final state

$$x_f = \begin{bmatrix} x_{1f} \\ x_{2f} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 50 \end{bmatrix}$$

The matrices (40) satisfy the condition (9).
Taking into account that

$$\Phi(t) = e^{A_{11} t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \quad (41)$$

and using (28) and (30), we obtain the monomial matrix

$$R_f = \int_0^{t_f} \Phi(\tau) B_1 B_1^T \Phi^T(\tau) d\tau = \int_0^1 \begin{bmatrix} e^{-2\tau} & 0 \\ 0 & e^{-4\tau} \end{bmatrix} d\tau = \begin{bmatrix} \frac{1 - e^{-2}}{2} & 0 \\ 0 & \frac{1 - e^{-4}}{4} \end{bmatrix} \quad (42)$$

and the input

$$u(t,0) = B_1^T \Phi(t_f - t) R_f^{-1} x_{1f}$$
$$= \begin{bmatrix} e^{t-1} & 0 \\ 0 & e^{2(t-1)} \end{bmatrix} \begin{bmatrix} \dfrac{1-e^{-2}}{2} & 0 \\ 0 & \dfrac{1-e^{-4}}{4} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \dfrac{2e^{-1}}{1-e^{-2}} e^t \\ \dfrac{4e^{-2}}{1-e^{-4}} e^{2t} \end{bmatrix}, \quad t \in [0,1] \tag{43}$$

From (29), (41) and (43) we have

$$\hat{x}_{2f} = x_{2f} - A_{21} \int_0^{t_f} \Phi(t_f - \tau) A_{12}$$
$$[A_{21} \int_0^{\tau} \Phi(\tau - \tau_1) B_1 u(\tau_1, 0) d\tau_1 + B_2 u(\tau, 0)] d\tau$$
$$-A_{22}[A_{21} x_{1f} + B_2 u(t_f, 0)]$$
$$= x_{2f} - \begin{bmatrix} 1 & 2 \end{bmatrix} \int_0^1 \begin{bmatrix} e^{\tau-1} \\ e^{2(\tau-1)} \end{bmatrix}$$
$$\left[\begin{bmatrix} 1 & 2 \end{bmatrix} \int_0^{\tau} \begin{bmatrix} e^{\tau_1 - \tau} & 0 \\ 0 & e^{2(\tau_1 - \tau)} \end{bmatrix} \begin{bmatrix} \dfrac{2e^{-1}}{1-e^{-2}} e^{\tau_1} \\ \dfrac{4e^{-2}}{1-e^{-4}} e^{2\tau_1} \end{bmatrix} d\tau_1 \right.$$
$$\left. + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \dfrac{2e^{-1}}{1-e^{-2}} e^{\tau} \\ \dfrac{4e^{-2}}{1-e^{-4}} e^{2\tau} \end{bmatrix} \right] d\tau$$
$$- [2] \left[\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \dfrac{2e^{-1}}{1-e^{-2}} e^{1} \\ \dfrac{4e^{-2}}{1-e^{-4}} e^{2} \end{bmatrix} \right]$$
$$= x_{2f} + \frac{1}{1-e^{-2}} \left(4,5e^{-2} - 7,5\right)$$
$$+ \frac{1}{1-e^{-4}} \left(5e^{-4} + 4e^{-3} - 21\right) - 6 \approx 15 \tag{44}$$

Note that the vector (44) satisfies the condition $\hat{x}_{2f} \in R_+^{n_2}$. In this case

$$P(t_f) = A_{21} \int_0^{t_f} \Phi(\tau) B_1 d\tau$$
$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1-e^{-1} & 0 \\ 0 & \frac{1-e^{-2}}{2} \end{bmatrix} \tag{45}$$
$$= \begin{bmatrix} 1-e^{-1} & 1-e^{-2} \end{bmatrix}.$$

Taking into account (44) and (45) it is easy to check that the condition (31) is met and equation (32) of the form

$$\begin{bmatrix} 2-e^{-1} & 3-e^{-2} \end{bmatrix} u_1 = \hat{x}_{2f}$$

has many solutions, for example

$$u_1 =$$
$$= \begin{bmatrix} \frac{e^{-8} + 30,5e^{-6} - 4e^{-5} - 33,5e^{-4} + 4e^{-3} - 14,5e^{-2} + 12,5}{(2-e^{-1})(1-e^{-2})(1-e^{-4})} \\ 1 \end{bmatrix}$$
$$\approx \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

Note that using (37) and (38), we obtain

$$W_f = \int_0^{t_f} \Phi(\tau) B_1 d\tau = (e^{A_{11} t_f} - I_{n_1}) A_{11}^{-1} B_1$$
$$= \begin{bmatrix} e^{-1} - 1 & 0 \\ 0 & e^{-2} - 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} e^{-1} - 1 & 0 \\ 0 & e^{-2} - 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -0,5 \end{bmatrix}$$
$$= \begin{bmatrix} 1 - e^{-1} & 0 \\ 0 & 0,5(1-e^{-2}) \end{bmatrix}$$

and

$$u_0 = W_f^{-1} x_{1f} = \begin{bmatrix} 1 - e^{-1} & 0 \\ 0 & 0,5(1-e^{-2}) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{1-e^{-1}} & 0 \\ 0 & \frac{1}{0,5(1-e^{-2})} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{1-e^{-1}} \\ \frac{1}{0,5(1-e^{-2})} \end{bmatrix}.$$

6. Concluding remarks

A new class of positive hybrid linear systems has been introduced. Necessary and sufficient condition for the positivity of the hybrid linear systems has been established. The classical Cayley-Hamilton theorem has been extended for the hybrid systems. The reachability of the hybrid systems has been defined and sufficient conditions for the reachability have been established. The considerations have been illustrated by numerical examples.

Extension of the presented reachability conditions for suitable controllability and observability conditions is possible but is not trivial [2,13].

An open problem is extension of the considerations for 2D hybrid systems described by models with structure similar to the Kurek model [2,15,16].

REFERENCES

[1] L. Farina and S. Rinaldi, *Positive Linear Systems; Theory and Applications*, J. Wiley, New York, 2000.

[2] T. Kaczorek, *Positive 1D and 2D systems*, Springer Verlag, London 2001.

[3] T. Kaczorek, "Some recent developments in positive systems", *Proc. 7th Conf. Dynamical Systems Theory and Applications*, 25–35 (2003).

[4] L. Benvenuti and L. Farina, "A tutorial on the positive realization problem", *IEEE Trans. Autom. Control* 49 (5), 651–664 (2004).

[5] T. Kaczorek, "A realization problem for positive continuous-time linear systems with reduced numbers of delay", *Int. J. Appl. Math. Comp. Sci.* 3, 325–331 (2006).

[6] T. Kaczorek, "Realization problem for positive multivariable discrete-time linear systems with delays in the state vector and inputs", *Int. J. Appl. Math. Comp. Sci.* 16 (2), 101–106 (2006).

[7] T. Kaczorek, "Realization problem for positive discrete-time systems with delay", *System Science* 30 (4), 17–30 (2004).

[8] T. Kaczorek, "Positive minimal realizations for singular discrete-time systems with delays in state and delays in control", *Bull. Pol. Ac.: Sci. Tech.* 53 (3), 293–298 (2005).

[9] T. Kaczorek and M. Busłowicz, "Minimal realization problem for positive multivariable linear systems with delay", *Int. J. Appl. Math. Comput. Sci.* 14 (2), 181–187 (2004).

[10] M. Busłowicz and T. Kaczorek, "Reachability and minimum energy control of positive linear discrete-time systems with one delay", *12th Mediterranean Conf. Control and Automation* 3, (2004).

[11] V.M. Marchenko and O.N. Poddubnaya, "Relative controllability of stationary hybrid systems", *10th IEEE Int. Conf. Methods and Models in Automation and Robotics*, 267–272 (2004).

[12] V.M. Marchenko, O.N. Poddubnaya, and Z. Zaczkiewicz, "On the observability of linear differential-algebraic systems with delays", *IEEE Trans. Autom. Contr.* 51 (8), 1387–1392 (2006).

[13] J. Klamka, *Controllability of Dynamical Systems*, Kluwer Academic Publ., Dordrecht, 1991.

[14] R.B. Roesser, "A discrete state-space model for linear image processing", *IEEE Trans. on Automatic Control* AC-20 (1), 1–10 (1975).

[15] J. Kurek, "The general state-space model for a two-dimensional linear digital system", *IEEE Trans. Autom. Contr.* AC-30, 600–602 (1985).

[16] M.E. Valcher, "On the initial stability and asymptotic behavior of 2D positive systems", *IEEE Trans. on Circuits and Systems* 44 (7), 602–613 (1997).