

# Dominance relations for two-machine flow shop problem with late work criterion

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**Abstract.** The paper concerns the two-machine non-preemptive flow shop scheduling problem with a total late work criterion and a common due date ( $F2|d_j = d|Y$ ). The late work performance measure estimates the quality of a solution with regard to the duration of late parts of activities performed in the system, not taking into account the quantity of this delay. In the paper, a few theorems are formulated and proven, describing features of an optimal solution for the problem mentioned, which is NP-hard. These theorems can be used in exact exponential algorithms (as dominance relations reducing the number of solutions enumerated explicitly), as well as in heuristic and metaheuristic methods (supporting the construction of sub-optimal schedules of a good quality).

**Key words:** scheduling, flow shop, late work criterion, dominance relation.

## 1. Introduction

The performance measure is a crucial component of the definition of any scheduling problem (cf. e.g. [1–3]). The viewpoint from which the quality of a schedule is estimated, i.e. the objective function, usually determines the strategy of searching for this solution. In most practical problems arising in different domains, the problem analysis has to take into account various time restrictions. This makes due date involving criteria, such as the late work one, especially interesting subject of the research.

The late work performance measure, investigated in this paper, evaluates schedules with regard to the number of tardy units of particular activities executed in a system. It can be considered as a special case of the imprecise computation model (cf. e.g. [4]), in which a job is divided in the mandatory and optional part. In a feasible schedule, the mandatory part of a job has to be early, while the optional part can be late, but the more its units is tardy the worse solution is obtained. The late work scheduling corresponds to the situation when the whole processing time is optional.

Such a concept of an objective function finds practical applications, for example, in the process of collecting data in control systems, where the amount of information exposed after a due date influences the accuracy of a steering algorithm. Moreover, the late work criterion can be considered in industrial systems, where late parts of customer orders should be minimized. It supports also production planning in a certain time horizon, since it allows reducing the amount of work not assigned to a particular time slot, which has to be considered in the following one. Furthermore, the late work concept can be applied in agriculture to optimize harvesting and land cultivation processes. In

the first case, the amount of wasted crop, in the latter one, the amount of not spread fertilizers or plant protection substances should be minimized.

The late work concept was introduced in the context of the scheduling problem on identical parallel machines [5] and, next, it was applied to uniform [6] and single [7–15] machine(s) cases. Then, the late work criteria were analyzed in the shop environment, i.e. in systems with dedicated machines [16–23], starting from the simplest two-machine models with a common due date. All cases with two machines and the weighted late work criterion appeared to be binary NP-hard [24] (i.e.  $O2|d_j = d|Y_w$  [17],  $F2|d_j = d|Y_w$  [18] and  $J2|d_j = d, n_j \leq 2|Y_w$  [19]). The two-machine open shop without weights ( $O2|d_j = d|Y$  [17]) is solvable in polynomial time. The two machine flow shop problem with a common due date and the total late work criterion,  $F2|d_j = d|Y$ , being the subject of this paper, has been recently classified as binary NP-hard. Its intractability results from the transformation from the partition problem [25]. Moreover, it is possible to apply pseudo-polynomial time dynamic programming developed for the weighted case  $F2|d_j = d|Y_w$  [18]. Since the job shop case is a generalization of the flow shop system (cf. e.g. [1–3]),  $J2|d_j = d, n_j \leq 2|Y_w$ , is also computationally hard.

In the paper, the features of an optimal solution for problem  $F2|d_j = d|Y$  are presented, which can be helpful in constructing efficient exact and heuristic approaches solving it. In Section 2, the formal definition of the case under consideration is given, together with the description of a general structure of an optimal solution of this problem. In Section 3, three dominance relations are formulated and proven, which support the process of construct-

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ing an optimal schedule for  $F2|d_j = d|Y$ . Conclusions and further research directions are given in Section 4.

## 2. Problem definition

The two machine flow shop problem with the late work criterion and a common due date,  $F2|d_j = d|Y$ , concerns scheduling of a set of jobs  $J = \{J_1, \dots, J_j, \dots, J_n\}$  on two dedicated machines  $M_1$  and  $M_2$ . Each job  $J_j$  has to be performed first on machine  $M_1$  and then on  $M_2$  for  $p_{1j}$  and  $p_{2j}$  time units, respectively. Hence, one can consider a job as a sequence of two tasks, for which a precedence constraint is defined. Each machine can process only one job at a time and each job can be executed by only one machine at a time. Moreover, a common due date  $d$  is defined for all jobs in the system. The goal is to construct a feasible non-preemptive schedule for which the total late work is minimal, i.e. the amount of work executed after a given common due date  $d$  is minimal.

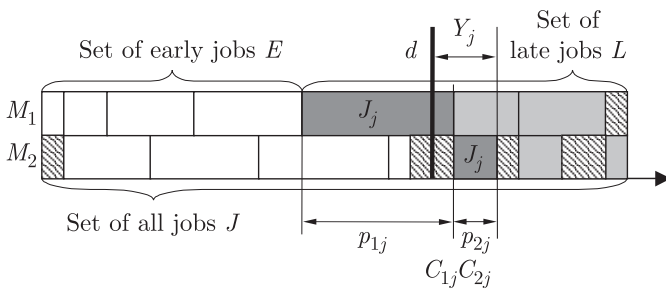


Fig. 1. The late work  $Y_j$  for job  $J_j$  in an exemplary solution for  $F2|d_j = d|Y$

Denoting by  $C_{ij}$  the completion time of job  $J_j$  on machine  $M_i$ , the late work  $Y_j$  for this job is determined as (cf. Fig. 1):

$$Y_j = \sum_{i=1,2} \min \{ \max \{ 0, C_{ij} - d \}, p_{ij} \}. \quad (1)$$

The criterion value to be minimized, estimating the quality of a complete schedule for the whole set of  $n$  jobs, is calculated as

$$Y = \sum_{j=1}^n Y_j. \quad (2)$$

The late work criterion represents slightly different point of view than other classical performance measures. If an activity is scheduled totally late, then the quantity of its delay does not influence the criterion value (on the contrary to the tardiness or lateness for example). Minimizing this performance measure, one tends to execute as many units of work before a common due date as possible, which is equivalent to minimizing idle time before  $d$  [18,23]. Such a strategy usually does not result in a schedule of the minimal length. Since early jobs are selected in order to fill the gap between time zero and  $d$  in the best way and jobs scheduled after a common due date are not important for the objective function and are executed in an arbitrary order, the whole solution can have quite large

makespan. On the contrary, all early activities have to be processed in the order that minimizes the schedule length [18,23]. Once it is decided which jobs are executed before a common due date, the set of such jobs should be executed as soon as possible, i.e. in Johnson's order [26].

Johnson's algorithm, designed for problem  $F2||C_{\max}$ , divides the set of jobs  $J$  into two subsets:  $J^1$  containing jobs  $J_j \in J$  with  $p_{1j} \leq p_{2j}$  and  $J^2$  built by jobs  $J_j \in J$  with  $p_{1j} > p_{2j}$ . In order to minimize the schedule length, jobs from set  $J^1$  should be sequenced in non-decreasing order of  $p_{1j}$  and precede jobs from  $J^2$  processed in non-increasing order of  $p_{2j}$ .

Summing up, solving problem  $F2|d_j = d|Y$ , one has to divide the set of all jobs  $J$  into two subsets  $J = E \cup L$  (cf. Fig. 1), where  $E$  contains early jobs, while late jobs (possibly partially) belong to  $L$ . Jobs from  $E$  have to be processed in Johnson's order, while the first late job in  $L$  (job  $J_j$  in Fig. 1) has to be chosen in the way minimizing idle time before  $d$ . The sequence of the remaining late jobs from set  $L$  is arbitrary.

Although the general structure of an optimal solution of problem  $F2|d_j = d|Y$ , described above, is quite strict, this scheduling case is NP-hard [24, 25]. The crucial decision, difficult from the viewpoint of the computational complexity, is to select the first late job in the system and, first of all, to divide the set of jobs into two subsets of early and late activities. In the paper, some dominance relations are formulated and proven, which support the process of constructing this division of the set of jobs.

## 3. Dominance relations

Since problem  $F2|d_j = d|Y$  is NP-hard, a polynomial time method solving it optimally is rather unlikely to exist [24]. Exact approaches of the exponential time complexity, such as a branch and bound algorithm for example, can be efficiently applied only for small problem instances. Exact methods have to explore the whole solution space in the search for an optimal schedule, but not every possible sequence of jobs is analyzed explicitly. The efficiency of such algorithms depends on their ability to discard partial solutions which do not lead to the optimal one. The careful analysis of the structure of an optimal schedule often results in formulating dominance relations, which justify discarding some parts of the solution space. On the other hand, these relations make it possible to design good heuristic or metaheuristic approaches, concentrating their search in the most promising areas of the problem space.

The optimal solution for problem  $F2|d_j = d|Y$  can be represented as a permutation of jobs. Thus, the solution space contains all permutations of the set of jobs. But, with regard to the fact that early jobs have to be executed in Johnson's order, all permutations of these activities which are not consistent with Johnson's order can be discarded. Obviously, the crucial decision is to select jobs to this set of early activities. Three theorems formulated

and proven in the paper impose conditions, which may be taken into account in this decision process. They show that selecting jobs in non-decreasing order of processing times on the first machine is usually profitable from the late work point of view.

**THEOREM 1.** In some optimal solution of problem  $F2|d_j = d|Y$ , there is no pair of jobs  $J_k, J_s \in J^1$  such that  $J_s \in E \wedge J_k \in L \wedge p_{1k} \leq p_{1s}$ .

**Proof.** Let assume that there exists an optimal solution  $\pi$ , for which the condition of Theorem 1 does not hold, i.e. there exist two jobs  $J_s, J_k \in J^1$  (i.e.  $p_{1s} \leq p_{2s}$  and  $p_{1k} \leq p_{2k}$ ) such that  $J_s \in E$  and  $p_{1k} \leq p_{1s}$ , but  $J_k \in L$ . Moreover, let assume, without the loss of generality, that  $J_s$  is the first job in the sequence of early jobs for which the theorem does not hold. There will be shown that processing job  $J_k$  early, one obtains a schedule  $\pi'$  satisfying the condition of Theorem 1 with the criterion value not worse than  $\pi$ .

If  $J_k$  is processed early, then it has to be executed directly before  $J_s$  in  $\pi'$ , due to the assumption that  $J_s$  is the first job contradicting Theorem 1, as well as to the fact that all early jobs are sequenced in Johnson's order ( $p_{1k} \leq p_{1s}$  and all jobs from  $J^1$ , including  $J_s$  and  $J_k$ , are scheduled in non-decreasing order of the processing times of their first tasks). Exemplary solutions  $\pi$  and  $\pi'$  are depicted in Fig. 2 (obviously, some partial schedules  $\pi_1, \pi_2, \pi_3$  can be empty, as well as some idle times can be equal to zero).

Because minimizing late work is equivalent to minimizing idle time before a common due date  $d$ , to prove the theorem, it is enough to show that the total idle time before  $d$  does not increase as a result of shifting  $J_k$  before  $J_s$  and that no unit of idle time is shifted to the left (especially before a due date  $d$ ).

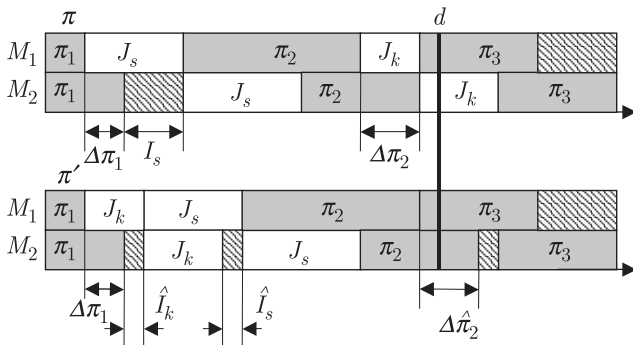


Fig. 2. Exemplary schedules  $\pi$  and  $\pi'$ , where  $J_k \in J^1$  is shifted before  $J_s \in J^1$  ( $p_{1k} \leq p_{1s}$ )

Let's denote with  $I_1, I_2, I_3, I_s$  the amount of idle time within partial schedules  $\pi_1, \pi_2, \pi_3$ , and before job  $J_s$  in schedule  $\pi$ , respectively (if there is idle time before  $J_k$  in  $\pi$ , then it is included in  $I_2$ ). The values  $\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{I}_k, \hat{I}_s$  denote the corresponding idle times in schedule  $\pi'$ . In order to avoid a possible violation of precedence constraints, partial schedules  $\pi_2$  and  $\pi_3$  cannot be shifted to the left

on  $M_2$  in  $\pi'$  with regard to  $\pi$  (cf. e.g. subschedule  $\pi_3$  in Fig. 2).

The total idle times for schedules  $\pi$  and  $\pi'$  are equal to  $I = I_1 + I_s + I_2 + I_3$  and  $\hat{I} = \hat{I}_1 + \hat{I}_k + \hat{I}_s + \hat{I}_2 + \hat{I}_3$ . In consequence, the change of the total idle time, representing the change of the criterion value, is equal to

$$\begin{aligned} \Delta I &= \hat{I} - I = (\hat{I}_1 + \hat{I}_2 + \hat{I}_3 + \hat{I}_k + \hat{I}_s) \\ &\quad - (I_1 + I_2 + I_3 + I_s) \\ &= (\hat{I}_1 - I_1) + (\hat{I}_2 - I_2) + (\hat{I}_3 - I_3) \\ &\quad + (\hat{I}_k + \hat{I}_s - I_s). \end{aligned} \quad (3)$$

Because the partial schedule  $\pi_1$  is the same in  $\pi$  as in  $\pi'$ , therefore  $(I_1 = \hat{I}_1) \Rightarrow (\hat{I}_1 - I_1 = 0)$  and

$$\Delta I = (\hat{I}_2 - I_2) + (\hat{I}_3 - I_3) + (\hat{I}_k + \hat{I}_s - I_s) \quad (4)$$

where

$$I_s = \max\{0, p_{1s} - \Delta\pi_1\}, \quad (5)$$

$$\hat{I}_k = \max\{0, p_{1k} - \Delta\pi_1\}, \quad (6)$$

$$\hat{I}_s = \max\{0, p_{1k} + p_{1s} - \max\{\Delta\pi_1, p_{1k}\} - p_{2k}\}. \quad (7)$$

Since it is assumed that  $\pi_2$  cannot be shifted to the left on  $M_2$ , the change of idle time for  $\pi_2$  is equal to the change of the difference between schedule lengths on machines  $M_1$  and  $M_2$  before  $\pi_2$  (i.e. the change of the schedule offset) in  $\pi'$  and  $\pi$ , this means that

$$\begin{aligned} (\hat{I}_2 - I_2) &= \{(p_{1k} + p_{1s}) - (\Delta\pi_1 + \hat{I}_k + p_{2k} + \hat{I}_s + p_{2s})\} \\ &\quad - \{(p_{1s}) - (\Delta\pi_1 + I_s + p_{2s})\} \\ &= p_{1k} - \hat{I}_k - p_{2k} - \hat{I}_s + I_s \\ &= (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s). \end{aligned} \quad (8)$$

First, jobs  $J_k, J_s$  and a partial schedule  $\pi_2$  will be considered, then a partial schedule  $\pi_3$  will be analyzed.

**Case 1.** Let's assume that

$$\hat{I}_k = 0 \Rightarrow p_{1k} - \Delta\pi_1 \leq 0 \Rightarrow p_{1k} \leq \Delta\pi_1. \quad (9)$$

Eq. (7) and (9) result in

$$\begin{aligned} \hat{I}_s &= \max\{0, p_{1k} + p_{1s} - \max\{\Delta\pi_1, p_{1k}\} - p_{2k}\} \\ &= \max\{0, p_{1k} + p_{1s} - \Delta\pi_1 - p_{2k}\} \\ &= \max\{0, (p_{1k} - p_{2k}) + (p_{1s} - \Delta\pi_1)\}. \end{aligned} \quad (10)$$

**Subcase 1.** Let's assume that

$$I_s = 0 \Rightarrow p_{1s} - \Delta\pi_1 \leq 0. \quad (11)$$

Taking into account (11) and  $J_k \in J^1 \Rightarrow p_{1k} - p_{2k} \leq 0$ , Eq. (10) reduces to  $\hat{I}_s = \max\{0, (p_{1k} - p_{2k}) + (p_{1s} - \Delta\pi_1)\} = 0$ . In consequence, taking into account (9) and (11), there is  $(\hat{I}_k + \hat{I}_s - \hat{I}_s) = 0$ . This means that no new idle time appears in a partial schedule before  $J_s$  in  $\pi'$ . Moreover, from (8) results  $(\hat{I}_2 - I_2) = (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - p_{2k}) \leq 0$ . It denotes that the schedule offset between  $M_1$  and  $M_2$  before  $\pi_2$  can only increase in  $\pi'$  in comparison to  $\pi$ , possibly reducing the internal idle time within  $\pi_2$ .

**Subcase 2.** Let's assume that

$$I_s > 0 \Rightarrow p_{1s} - \Delta\pi_1 > 0. \quad (12)$$

Eq. (5) and (12) result in

$$I_s = p_{1s} - \Delta\pi_1, \quad (13)$$

while (10) and (13) lead to

$$\begin{aligned} \hat{I}_s &= \max\{0, (p_{1k} - p_{2k}) + (p_{1s} - \Delta\pi_1)\} \\ &= \max\{0, p_{1k} - p_{2k} + I_s\}. \end{aligned} \quad (14)$$

First, let's assume that

$$\hat{I}_s = 0 \Rightarrow p_{1k} - p_{2k} + I_s \leq 0. \quad (15)$$

Equations (9), (12) and (15) lead to  $(\hat{I}_k + \hat{I}_s - I_s) = I_s \leq 0$  and Eq. (8) reduces to  $(\hat{I}_2 - I_2) = (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) = p_{1k} - p_{2k} + I_s \leq 0$ .

Then, let's assume that

$$\hat{I}_s > 0 \Rightarrow p_{1k} - p_{2k} + I_s > 0. \quad (16)$$

Equations (14) and (16) denote that  $\hat{I}_s = p_{1k} - p_{2k} + I_s$ . Combining this observation with (9) and then with (8), one obtains  $(\hat{I}_k + \hat{I}_s - I_s) = 0 + (p_{1k} - p_{2k} + I_s) - I_s = p_{1k} - p_{2k} \leq 0$  and  $(\hat{I}_2 - I_2) = (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - p_{2k}) - (p_{1k} - p_{2k}) = 0$ , respectively.

Regardless of the value  $\hat{I}_s$  in Subcase 2, idle time in a partial schedule before  $J_s$  cannot increase, because job  $J_k$  may only reduce idle time before  $J_s$  ( $\hat{I}_k + \hat{I}_s - I_s \leq 0$ ). Moreover, the schedule offset before  $\pi_2$  cannot become smaller in  $\pi'$  than in  $\pi$  ( $(\hat{I}_2 - I_2) \leq 0$ ), and, in consequence, the internal idle time within  $\pi_2$  cannot increase (actually, it can be possibly reduced, if  $(\hat{I}_2 - I_2) < 0$ ).

**Case 2.** Let's assume that

$$\hat{I}_k > 0 \Rightarrow p_{1k} - \Delta\pi_1 > 0 \Rightarrow p_{1k} > \Delta\pi_1. \quad (17)$$

Eq. (6) and (17) imply

$$\hat{I}_k = p_{1k} - \Delta\pi_1. \quad (18)$$

Because  $p_{1s} \geq p_{1k}$ , from Eq. (17) there is  $p_{1s} > \Delta\pi_1$  and from (5)

$$I_s = \max\{0, p_{1s} - \Delta\pi_1\} = p_{1s} - \Delta\pi_1 > 0. \quad (19)$$

From Eq. (7) and (17), there is

$$\begin{aligned} \hat{I}_s &= \max\{0, p_{1k} + p_{1s} - \max\{\Delta\pi_1, p_{1k}\} - p_{2k}\} \\ &= \max\{0, p_{1k} + p_{1s} - p_{1k} - p_{2k}\} \\ &= \max\{0, p_{1s} - p_{2k}\}. \end{aligned} \quad (20)$$

**Subcase 1.** Let's assume that

$$\hat{I}_s = 0 \Rightarrow p_{1s} - p_{2k} \leq 0. \quad (21)$$

Eq. (18), (19) and (21) lead to  $(\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - \Delta\pi_1) - (p_{1s} - \Delta\pi_1) = p_{1k} - p_{1s} \leq 0$  and reduce (8) to  $(\hat{I}_2 - I_2) = (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - p_{2k}) - (p_{1k} - p_{1s}) = p_{1s} - p_{2k} \leq 0$ .

**Subcase 2.** Let's assume that

$$\hat{I}_s > 0 \Rightarrow p_{1s} - p_{2k} > 0 \Rightarrow \hat{I}_s = p_{1s} - p_{2k}. \quad (22)$$

Eq. (18), (19) and (22) lead to  $(\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - \Delta\pi_1) + (p_{1s} - p_{2k}) - (p_{1s} - \Delta\pi_1) = p_{1k} - p_{2k} \leq 0$  and reduce (8) to  $(\hat{I}_2 - I_2) = (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - p_{2k}) - (p_{1k} - p_{2k}) = 0$ .

Again, in both subcases of Case 2, idle time in a schedule before  $J_s$  cannot increase ( $\hat{I}_k + \hat{I}_s - I_s \leq 0$ ) and the schedule offset before  $\pi_2$  cannot decrease in  $\pi'$  with regard to  $\pi$  ( $(\hat{I}_2 - I_2) \leq 0$ ).

To complete the proof, a partial schedule  $\pi_3$  will be considered. From the case study presented above it results that  $(\hat{I}_2 - I_2) \leq 0$ . This means that the schedule offset between  $M_1$  and  $M_2$  before  $\pi_2$  can only increase in  $\pi'$  with regard to  $\pi$  or it remains unchanged. In the latter case the schedule offset after  $\pi_2$  does not change too,  $\Delta\pi_2 = \Delta\hat{\pi}_2$ . In the first case, the larger offset before  $\pi_2$  may reduce the internal idle time in this subschedule (if any) and, then, increase the offset between machine  $M_1$  and  $M_2$  after  $\pi_2$ ,  $\Delta\pi_2 \leq \Delta\hat{\pi}_2$ . Summing up, there is  $\Delta\pi_2 \leq \Delta\hat{\pi}_2$ .

Similarly, as for  $\pi_2$ , a partial schedule  $\pi_3$  cannot be shifted to the left on  $M_2$  in  $\pi'$  and, in consequence,  $(\hat{I}_3 - I_3)$  is equal to the change of the difference in the schedule offsets between machines  $M_1$  and  $M_2$  in  $\pi'$  and  $\pi$ , i.e.:

$$\begin{aligned} (\hat{I}_3 - I_3) &= \{-\Delta\hat{\pi}_2\} - \{(p_{1k}) - (\Delta\pi_2 + p_{2k})\} \\ &= -(p_{1k} - p_{2k}) + \Delta\pi_2 - \Delta\hat{\pi}_2. \end{aligned} \quad (23)$$

Combining (23) with (8), the total idle time change (4) is equal to

$$\begin{aligned} \Delta I &= (\hat{I}_2 - I_2) + (\hat{I}_3 - I_3) + (\hat{I}_k + \hat{I}_s - I_s) \\ &= (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) + (\hat{I}_3 - I_3) \\ &\quad + (\hat{I}_k + \hat{I}_s - I_s) \\ &= (p_{1k} - p_{2k}) + (\hat{I}_3 - I_3) \\ &= (p_{1k} - p_{2k}) - (p_{1k} - p_{2k}) + \Delta\pi_2 - \Delta\hat{\pi}_2 \\ &= \Delta\pi_2 - \Delta\hat{\pi}_2 \leq 0. \end{aligned} \quad (24)$$

This means that the total idle time cannot increase in consequence of executing job  $J_k$  before  $J_s$ . The possible idle time which may appear in  $\pi'$  before a subschedule  $\pi_3$  (if  $(\hat{I}_3 - I_3) > 0$ , cf. Fig. 2) is a result of shifting some idle time units to the right. The total number of idle time units does not increase; some of them can be shifted from the left to the right part of a schedule. If this shift is large enough, some idle time units can be moved after a common due date, causing an additional decrease of the late work in the system.

Summing up, it is shown that  $\Delta I \leq 0$  and no unit of idle time is shifted to the left as a result of a schedule modification. Executing job  $J_k$  before  $J_s$  leads to the new schedule  $\pi'$ , which is not worse than  $\pi$  with regard to the total late work value. This analysis can be repeated for the remaining pairs of jobs  $J_s, J_k \in J^1$  such that  $J_s \in E \wedge J_k \in L \wedge p_{1k} \leq p_{1s}$  proving Theorem 1 (note that Theorem 1 holds, if  $J_s$  is early as well as it is late in  $\pi'$ ).

**THEOREM 2.** In some optimal solution of problem  $F2|d_j = d|Y$ , there is no pair of jobs  $J_k \in J^1, J_s \in J^2$  such that  $J_s \in E \wedge J_k \in L \wedge p_{1k} \leq p_{1s}$ .

**Proof.** Let's assume that there exists an optimal solution  $\pi$ , for which the condition of Theorem 2 does not

hold, i.e. there exist two jobs  $J_s \in J^2$  (i.e.  $p_{1s} > p_{2s}$ ),  $J_k \in J^1$  (i.e.  $p_{1k} \leq p_{2k}$ ), such that  $J_s \in E$  and  $p_{1k} \leq p_{1s}$ , but  $J_k \in L$ . Moreover, let assume without the loss of generality, that  $J_s$  is the first job in the sequence of early jobs for which the theorem does not hold. It will be shown that processing job  $J_k$  early, one obtains a schedule  $\pi'$  satisfying the condition of Theorem 2 with the criterion value not worse than  $\pi$ . Taking into account the assumptions that  $J_k \in J^1$  and  $J_s \in J^2$ , if  $J_k$  is early, then it has to be processed before  $J_s$  in  $\pi'$ , because all early jobs from set  $J^1$  (including  $J_k$ ) precede all early jobs from set  $J^2$  (including  $J_s$ ) in Johnson's order. Exemplary solutions  $\pi$  and  $\pi'$  are depicted in Fig. 3 (obviously, some partial schedules  $\pi_1, \pi_2, \pi_3, \pi_4$  can be empty as well as some idle times can be equal to zero). To prove the theorem, it is enough to show that the total idle time does not increase as a result of shifting  $J_k$  before  $J_s$  and that no unit of idle time is shifted to the left (before a common due date  $d$ ).

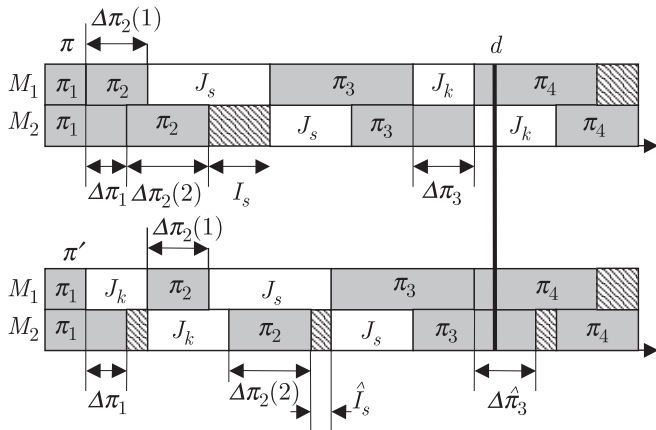


Fig. 3. Exemplary schedules  $\pi$  and  $\pi'$ , where  $J_k \in J^1$  is shifted before  $J_s \in J^2$  ( $p_{1k} \leq p_{1s}$ )

Let's denote with  $I_1, I_2, I_3, I_4, I_s$  the amount of idle time within partial schedules  $\pi_1, \pi_2, \pi_3, \pi_4$  and before job  $J_s$  in schedule  $\pi$  (a possible idle time before  $J_k$  is included in  $I_3$ ). The values  $\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{I}_4, \hat{I}_s, \hat{I}_k$  denote the corresponding idle times in schedule  $\pi'$ . The total idle times for schedules  $\pi$  and  $\pi'$  are equal to  $I = I_1 + I_s + I_2 + I_3 + I_4$  and  $\hat{I} = \hat{I}_1 + \hat{I}_k + \hat{I}_2 + \hat{I}_s + \hat{I}_3 + \hat{I}_4$  respectively. In consequence, the total idle time difference between  $\pi$  and  $\pi'$  is equal to:

$$\begin{aligned} \Delta I &= (\hat{I}_1 - I_1) + (\hat{I}_2 - I_2) + (\hat{I}_3 - I_3) \\ &\quad + (\hat{I}_4 - I_4) + (\hat{I}_k + \hat{I}_s - I_s). \end{aligned} \quad (25)$$

Because the partial schedule  $\pi_1$  does not change in  $\pi'$  in comparison to  $\pi$ , there is  $(I_1 = \hat{I}_1) \Rightarrow (\hat{I}_1 - I_1 = 0)$  and Eq. (25) leads to

$$\begin{aligned} \Delta I &= (\hat{I}_2 - I_2) + (\hat{I}_3 - I_3) + (\hat{I}_4 - I_4) \\ &\quad + (\hat{I}_k + \hat{I}_s - I_s). \end{aligned} \quad (26)$$

Similarly as in the proof for Theorem 1, in order to avoid a possible violation of precedence constraints within jobs, the partial schedules  $\pi_2, \pi_3, \pi_4$  cannot be shifted to

the left on  $M_2$  in  $\pi'$  with regard to  $\pi$  (cf. e.g. subschedule  $\pi_4$  in Fig. 3). In consequence, the possible change in idle time within  $\pi_2$ ,  $(\hat{I}_2 - I_2)$ , is equal to the change of the offsets between schedules on machines  $M_1$  and  $M_2$  in  $\pi'$  and  $\pi$ , respectively, i.e.:

$$\begin{aligned} (\hat{I}_2 - I_2) &= \{p_{1k} - (\Delta\pi_1 + \hat{I}_k + p_{2k})\} - \{-\Delta\pi_1\} \\ &= p_{1k} - p_{2k} - \hat{I}_k. \end{aligned} \quad (27)$$

Taking into account the fact that  $J_k \in J^1 \Rightarrow p_{1k} \leq p_{2k} \Rightarrow p_{1k} - p_{2k} \leq 0$ , as well as the fact that an idle time value cannot be negative,  $\hat{I}_k \geq 0 \Rightarrow -\hat{I}_k \leq 0$ , Eq. (27) and (26) lead to

$$(\hat{I}_2 - I_2) \leq 0, \quad (28)$$

$$\Delta I \leq (\hat{I}_3 - I_3) + (\hat{I}_4 - I_4) + (\hat{I}_k + \hat{I}_s - I_s). \quad (29)$$

Equation (28) implies that the offset between partial schedules on  $M_1$  and  $M_2$  before  $\pi_2$  cannot decrease in  $\pi'$  in comparison to  $\pi$ . In consequence, no additional idle time appears before the first job of  $\pi_2$  in  $\pi'$  and the internal idle time within  $\pi_2$  (if any) can be only reduced.

Moreover, the change of the schedule offset before a subschedule  $\pi_3$ , i.e. the possible idle time change for  $\pi_3$ , can be determined as

$$\begin{aligned} (\hat{I}_3 - I_3) &= \{(p_{1k} + \pi_2(1) + p_{1s}) \\ &\quad - (\Delta\pi_1 + \hat{I}_k + p_{2k} + \pi_2(2) + \hat{I}_s + p_{2s})\} \\ &\quad - \{(\pi_2(1) + p_{1s}) - (\Delta\pi_1 + \pi_2(2) + I_s + p_{2s})\} \\ &= (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s), \end{aligned} \quad (30)$$

where  $\pi_2(i)$  denotes the length of a partial schedule  $\pi_2$  on machine  $M_i$  and

$$\hat{I}_k = \max\{0, p_{1k} - \Delta\pi_1\}, \quad (31)$$

$$\begin{aligned} \hat{I}_s &= \max\{0, p_{1k} + \pi_2(1) + p_{1s} - \Delta\pi_1 - \hat{I}_k \\ &\quad - p_{2k} - \pi_2(2)\} \\ &= \max\{0, (p_{1k} - p_{2k}) - \hat{I}_k + (\pi_2(1) + p_{1s} \\ &\quad - \Delta\pi_1 - \pi_2(2))\}, \end{aligned} \quad (32)$$

$$I_s = \max\{0, \pi_2(1) + p_{1s} - \Delta\pi_1 - \pi_2(2)\}. \quad (33)$$

To prove that the change of the schedule offset before  $\pi_3$ ,  $(\hat{I}_3 - I_3)$ , cannot be positive and to show that no unit of idle time is shifted to the left, the following two cases have to be considered.

**Case 1.** Let's assume that

$$\hat{I}_k = 0 \Rightarrow p_{1k} - \Delta\pi_1 \leq 0 \Rightarrow p_{1k} \leq \Delta\pi_1. \quad (34)$$

Eq. (34) and (32) imply

$$\begin{aligned} \hat{I}_s &= \max\{0, (p_{1k} - p_{2k}) - \hat{I}_k + (\pi_2(1) \\ &\quad + p_{1s} - \Delta\pi_1 - \pi_2(2))\} \\ &= \max\{0, (p_{1k} - p_{2k}) + (\pi_2(1) + p_{1s} \\ &\quad - \Delta\pi_1 - \pi_2(2))\}. \end{aligned} \quad (35)$$

**Subcase 1.** Let's assume that

$$I_s = 0 \Rightarrow \pi_2(1) + p_{1s} - \Delta\pi_1 - \pi_2(2) \leq 0. \quad (36)$$

From Eq. (36) and  $p_{1k} - p_{2k} \leq 0$ , Eq. (35) reduces to  $\hat{I}_s = \max\{0, (p_{1k} - p_{2k}) + (\pi_2(1) + p_{1s} - \Delta\pi_1 - \pi_2(2))\} = 0$ . Combining this observation with (34) and (36), one obtains  $(\hat{I}_k + \hat{I}_s - I_s) = 0$ , and Eq. (30) reduces to  $(\hat{I}_3 - I_3) = (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - p_{2k}) \leq 0$ .

**Subcase 2.** Let's assume that

$$I_s > 0 \Rightarrow \pi_2(1) + p_{1s} - \Delta\pi_1 - \pi_2(2) > 0. \quad (37)$$

Equations (33) and (37) denote that

$$I_s = \pi_2(1) + p_{1s} - \Delta\pi_1 - \pi_2(2), \quad (38)$$

while Eq. (35) and (38) lead to

$$\begin{aligned} \hat{I}_s &= \max\{0, (p_{1k} - p_{2k}) + (\pi_2(1) + p_{1s} - \Delta\pi_1 - \pi_2(2))\} \\ &= \max\{0, (p_{1k} - p_{2k}) + I_s\}. \end{aligned} \quad (39)$$

First, let's assume that

$$\hat{I}_s = 0 \Rightarrow (p_{1k} - p_{2k}) + I_s \leq 0. \quad (40)$$

Eq. (34), (37) and (40) lead to  $(\hat{I}_k + \hat{I}_s - I_s) = 0 + 0 - I_s < 0$ , and reduce Eq. (30) to  $(\hat{I}_3 - I_3) = (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - p_{2k}) + I_s \leq 0$ .

Then, let's assume that

$$\hat{I}_s > 0 \Rightarrow (p_{1k} - p_{2k}) + I_s > 0, \quad (41)$$

which is equivalent to  $\hat{I}_s = (p_{1k} - p_{2k}) + I_s$ . This observation combined with Eq. (34) results in  $(\hat{I}_k + \hat{I}_s - I_s) = 0 + (p_{1k} - p_{2k} + I_s) - I_s = p_{1k} - p_{2k} \leq 0$ , and with Eq. (30) in  $(\hat{I}_3 - I_3) = (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - p_{2k}) - (p_{1k} - p_{2k}) = 0$ .

All subcases investigated within Case 1, showed that  $(\hat{I}_3 - I_3) \leq 0$  and  $(\hat{I}_k + \hat{I}_s - I_s) \leq 0$ . Since  $(\hat{I}_3 - I_3) \leq 0$ , the schedule offset before  $\pi_3$  can only increase (or remain the same) in  $\pi'$  with regard to  $\pi$ , possibly reducing the internal idle time within  $\pi_3$ . Moreover, no new idle time appears between time zero and the end of subschedule  $\pi_3$  in  $\pi'$ , because in Eq. (25) there is  $(\hat{I}_1 - I_1) = 0$ ,  $(\hat{I}_2 - I_2) \leq 0$ ,  $(\hat{I}_3 - I_3) \leq 0$  and  $(\hat{I}_k + \hat{I}_s - I_s) \leq 0$ .

**Case 2.** Let's assume that

$$\hat{I}_k > 0 \Rightarrow p_{1k} - \Delta\pi_1 > 0 \Rightarrow p_{1k} > \Delta\pi_1. \quad (42)$$

Equations (31) and (42) imply

$$\hat{I}_k = p_{1k} - \Delta\pi_1. \quad (43)$$

**Subcase 1.** Let's assume that

$$I_s = 0 \Rightarrow \pi_2(1) + p_{1s} - \Delta\pi_1 - \pi_2(2) \leq 0. \quad (44)$$

Taking into account (42), (44) and  $p_{1k} - p_{2k} \leq 0$ , Eq. (32) reduces to  $\hat{I}_s = 0$  and, in consequence, Eq. (30) is reduced to  $(\hat{I}_3 - I_3) = (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - p_{2k}) - (\hat{I}_k + 0 - 0) \leq 0$ . This means that the schedule offset before  $\pi_3$  cannot decrease in  $\pi'$  with regard to  $\pi$ , there is rather possible a reduction of an internal idle time within  $\pi_3$ . Moreover, idle time between time zero and the end of subschedule  $\pi_3$  is not larger in  $\pi'$  than in  $\pi$ , because  $(\hat{I}_1 - I_1) + (\hat{I}_2 - I_2) + (\hat{I}_k + \hat{I}_s - I_s) + (\hat{I}_3 - I_3) = 0 + (p_{1k} - p_{2k} - \hat{I}_k) + (\hat{I}_k + 0 - 0) + (\hat{I}_3 - I_3) = (p_{1k} - p_{2k}) + (\hat{I}_3 - I_3) \leq 0$ .

**Subcase 2.** Let's assume that

$$I_s > 0 \Rightarrow \pi_2(1) + p_{1s} - \Delta\pi_1 - \pi_2(2) > 0. \quad (45)$$

Equations (33) and (45) imply

$$I_s = \pi_2(1) + p_{1s} - \Delta\pi_1 - \pi_2(2), \quad (46)$$

while Eq. (32) and (46) result in

$$\begin{aligned} \hat{I}_s &= \max\{0, (p_{1k} - p_{2k}) - \hat{I}_k + (\pi_2(1) + p_{1s} \\ &\quad - \Delta\pi_1 - \pi_2(2))\} \\ &= \max\{0, (p_{1k} - p_{2k}) - \hat{I}_k + I_s\}. \end{aligned} \quad (47)$$

First, let's assume that

$$\hat{I}_s = 0 \Rightarrow (p_{1k} - p_{2k}) + I_s - \hat{I}_k \leq 0. \quad (48)$$

Equations (30) and (43), (46) and (48) result in

$$\begin{aligned} (\hat{I}_3 - I_3) &= (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) \\ &= (p_{1k} - p_{2k}) - (\hat{I}_k + 0 - I_s) \\ &= (p_{1k} - p_{2k}) - (p_{1k} - \Delta\pi_1) \\ &\quad + (\pi_2(1) + p_{1s} - \Delta\pi_1 - \pi_2(2)) \\ &= \pi_2(1) + p_{1s} - p_{2k} - \pi_2(2). \end{aligned} \quad (49)$$

Because according to Eq. (42) there is idle time before  $J_k$  in  $\pi'$  ( $\hat{I}_k > 0$ ), Eq. (49) determines idle time before  $J_s$  in  $\pi'$ , i.e.  $\hat{I}_s$  (note, that there is assumed that a partial schedule  $\pi_2$  cannot be shifted to the left on  $M_2$  in  $\pi'$  in comparison to  $\pi$ ). Eq. (48) states that  $\hat{I}_s = 0$ , and in consequence  $(\hat{I}_3 - I_3) = \hat{I}_s = 0$ . This means that the schedule offset before  $\pi_3$  is the same in  $\pi'$  as in  $\pi$ . Moreover, idle time between time zero and the end of subschedule  $\pi_3$  is not larger in  $\pi'$  than in  $\pi$ , because  $(\hat{I}_1 - I_1) + (\hat{I}_2 - I_2) + (\hat{I}_k + \hat{I}_s - I_s) + (\hat{I}_3 - I_3) = 0 + (p_{1k} - p_{2k} - \hat{I}_k) + (\hat{I}_k + 0 - I_s) + 0 = p_{1k} - p_{2k} - I_s \leq 0$ .

Now, let's assume that

$$\hat{I}_s > 0 \Rightarrow p_{1k} - p_{2k} + I_s - \hat{I}_k > 0. \quad (50)$$

Equations (47) and (50) imply that

$$\hat{I}_s = p_{1k} - p_{2k} + I_s - \hat{I}_k. \quad (51)$$

Equations (51) results in  $(\hat{I}_k + \hat{I}_s - I_s) = \hat{I}_k + p_{1k} - p_{2k} + I_s - \hat{I}_k - I_s = p_{1k} - p_{2k} \leq 0$ , and Eq. (30) reduces to  $(\hat{I}_3 - I_3) = (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) = (p_{1k} - p_{2k}) - (p_{1k} - p_{2k}) = 0$ . In consequence, the idle time change between time zero and the end of subschedule  $\pi_3$  can be estimated as  $(\hat{I}_1 - I_1) + (\hat{I}_2 - I_2) + (\hat{I}_k + \hat{I}_s - I_s) + (\hat{I}_3 - I_3) \leq 0$ .

All subcases investigated within Case 2, showed that the schedule offset before  $\pi_3$  cannot decrease in  $\pi'$  with regard to  $\pi$ , because  $(\hat{I}_3 - I_3) \leq 0$ . Actually, the internal idle time within  $\pi_3$  may become smaller, if  $(\hat{I}_3 - I_3) < 0$ . Moreover, idle time between time zero and the end of subschedule  $\pi_3$  in  $\pi'$  can be only reduced (or it remains the same) in  $\pi'$  in comparison to  $\pi$ , because  $(\hat{I}_1 - I_1) + (\hat{I}_2 - I_2) + (\hat{I}_k + \hat{I}_s - I_s) + (\hat{I}_3 - I_3) \leq 0$ .

Finally, it is necessary to show that the proposed modification of a schedule does not increase the criterion value within a partial schedule  $\pi_4$ .

Equations (29) and (30) result in

$$\begin{aligned} \Delta I &\leq (p_{1k} - p_{2k}) - (\hat{I}_k + \hat{I}_s - I_s) + (\hat{I}_4 - I_4) \\ &\quad + (\hat{I}_k + \hat{I}_s - I_s) \\ &= (p_{1k} - p_{2k}) + (\hat{I}_4 - I_4). \end{aligned} \tag{52}$$

Similarly as for  $\pi_2$ , it is proven that the offset between partial schedules on  $M_1$  and  $M_2$  before  $\pi_3$  cannot decrease (because  $(\hat{I}_3 - I_3) \leq 0$ ) and, in consequence, no additional idle time before the first job of  $\pi_3$  in  $\pi'$  appears. The offset increase before  $\pi_3$  can only reduce an internal idle time within this subschedule (if any) or make the difference between the schedule lengths on  $M_1$  and  $M_2$  after  $\pi_3$  larger in  $\pi'$  ( $\Delta\hat{\pi}_3$ ) than in  $\pi$  ( $\Delta\pi_3$ ), i.e.  $\Delta\hat{\pi}_3 \geq \Delta\pi_3$ .

In consequence, the change of the schedule offset before  $\pi_4$ , equivalent to the change of idle time, can be calculated as (note, that if there is an idle time before  $J_k$  in  $\pi$ , then it is included in  $I_3$ ):

$$\begin{aligned} (\hat{I}_4 - I_4) &= (-\Delta\hat{\pi}_3) - (p_{1k} - \Delta\pi_3 - p_{2k}) \\ &= (\Delta\pi_3 - \Delta\hat{\pi}_3) - (p_{1k} - p_{2k}). \end{aligned} \tag{53}$$

Applying (53) to Eq. (52), one obtains

$$\begin{aligned} \Delta I &\leq (p_{1k} - p_{2k}) + (\hat{I}_4 - I_4) \\ &= (p_{1k} - p_{2k}) + (\Delta\pi_3 - \Delta\hat{\pi}_3) - (p_{1k} - p_{2k}) \\ &= \Delta\pi_3 - \Delta\hat{\pi}_3 \leq 0. \end{aligned} \tag{54}$$

Summing up, the total idle time in the system cannot increase,  $\Delta I \leq 0$ , and no unit of idle time is shifted to the left as a result of a schedule modification (actually, some idle time can be shifted to the right, before  $\pi_4$ , if  $\hat{I}_4 - I_4 > 0$ , cf. Fig. 3). Shifting job  $J_k$  before  $J_s$  leads to the schedule  $\pi'$  not worse than  $\pi$  with regard to the total late work value. The repeated analysis of the remaining pairs of jobs  $J_s \in J^2$ ,  $J_k \in J^1$  such that  $J_s \in E \wedge J_k \in L \wedge p_{1k} \leq p_{1s}$ , proves Theorem 2 (note that Theorem 2 holds if  $J_s$  is early as well as it is late in  $\pi'$ ).

Theorems 1 and 2 state that, if one decides to execute a certain job  $J_s \in J$  early, then executing all jobs  $J_k \in J^1$  with  $p_{1k} \leq p_{1s}$  early leads to a schedule not worse than an original one.

Unfortunately, this rule cannot be applied for jobs  $J_k \in J^2$  straightforwardly. If there is a solution with two jobs  $J_s \in E$  and  $J_k \in J^2 \cap L$ , and  $J_k$  precedes  $J_s$  in Johnson's order,  $J_k \rightarrow J_s$ , (where  $J_k \in J^2$  and  $J_k \rightarrow J_s$  imply that  $J_s \in J^2$  and  $p_{2s} \leq p_{2k}$ ), then introducing job  $J_k$  into a sequence of early jobs may lead to a schedule worse than an original one. The quality of a schedule may deteriorate, if there is idle time before  $J_s$  in an original sequence (cf. Fig. 4), as well as if there is no idle time before this job (cf. Fig. 5).

Nevertheless, for such a pair of jobs  $J_k, J_s \in J^2$  and  $p_{2s} \leq p_{2k}$  a dominance relation analogous to Theorems 1 and 2 can be still formulated. In the case of activities from set  $J^2$  it is profitable to interchange job  $J_s$  with  $J_k$  (Theorem 3) instead of shifting  $J_k$  before  $J_s$  (cf. Theorems 1 and 2).

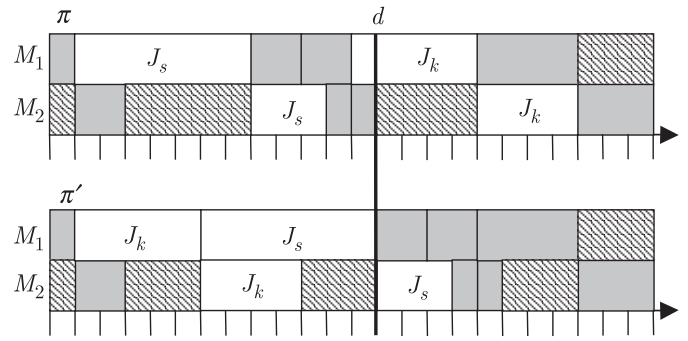


Fig. 4. Exemplary schedules  $\pi$  and  $\pi'$ , where  $J_k \in J^2$  is shifted before  $J_s \in J^2 (p_{1k} \leq p_{1s})$  and there is idle time before  $J_s$  in  $\pi$

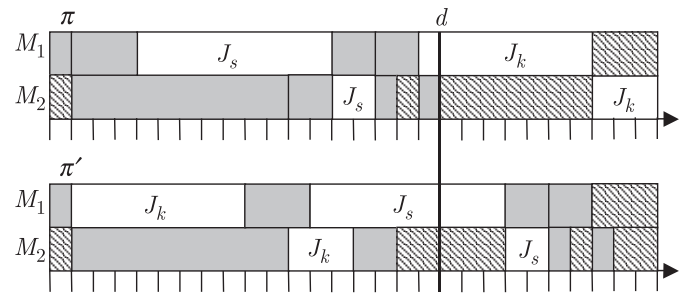


Fig. 5. Exemplary schedules  $\pi$  and  $\pi'$ , where  $J_k \in J^2$  is shifted before  $J_s \in J^2 (p_{1k} \leq p_{1s})$  and there is no idle time before  $J_s$  in  $\pi$

**THEOREM 3.** In some optimal solution of problem  $F2|d_j = d|Y$ , there is no pair of jobs  $J_k, J_s \in J^2$  such that  $J_s \in E \wedge J_k \in L \wedge p_{1k} \leq p_{1s} \wedge p_{2s} \leq p_{2k}$ .

**Proof.** Let's assume that there exists an optimal solution  $\pi$ , for which the condition of Theorem 3 does not hold, i.e. there exist two jobs  $J_k, J_s \in J^2$ , such that  $J_s \in E$ ,  $p_{1k} \leq p_{1s}$ ,  $p_{2s} \leq p_{2k}$ , but  $J_k \in L$ . Moreover, let assume that  $J_s$  is the first such a job in a sequence of early jobs. It will be shown that interchanging job  $J_s$  with  $J_k$  leads to a schedule  $\pi'$  satisfying the condition of Theorem 3 with the criterion value not worse than  $\pi$ .

With regard to the assumption that  $J_s$  is the first job contradicting Theorem 3, as well as to the fact that all early jobs are sequenced in Johnson's order, if  $J_k$  is early, then it starts in  $\pi'$  not later than  $J_s$  in  $\pi$  (all early jobs from  $J^2$ , are scheduled in non-increasing order of processing times of second tasks and  $p_{2s} \leq p_{2k}$ ). There are two cases possible, that after interchanging job  $J_s$  with  $J_k$ , job  $J_k$  starts at the same time in  $\pi'$  as  $J_s$  in  $\pi$  (i.e. there is no early job from  $J^2$  that is processed before  $J_s$  in  $\pi$  and after  $J_k$  in  $\pi'$ , cf. Fig. 6) or  $J_k$  starts earlier in  $\pi'$  than  $J_s$  in  $\pi$  (i.e. there are some early jobs from  $J^2$  that are processed before  $J_s$  in  $\pi$  and after  $J_k$  in  $\pi'$ , cf. Fig. 7).

**Case 1.** Let's assume that  $J_s$  in  $\pi$  and  $J_k$  in  $\pi'$  starts at the same time. Exemplary solutions  $\pi$  and  $\pi'$  are depicted in Fig. 6. Some partial schedules  $\pi_1, \pi_3, \pi_4$  can be empty as well as some idle times  $I_s, I_k, \hat{I}_s$  and  $\hat{I}_k$  can be equal to zero.

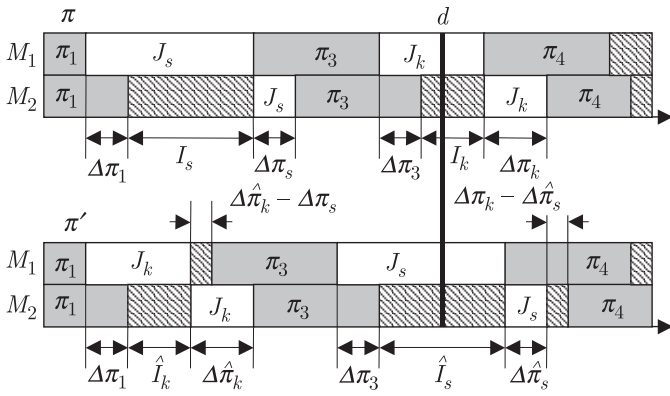


Fig. 6. Exemplary schedules  $\pi$  and  $\pi'$ , where  $J_k \in J^2$  is interchanged with  $J_s \in J^2$  ( $p_{1k} \leq p_{1s} \wedge p_{2s} \leq p_{2k}$ ) and there is no job executed before  $J_s$  in  $\pi$  and after  $J_k$  in  $\pi'$

Let's assume at the beginning that the structures of subsequences  $\pi_3$  and  $\pi_4$  do not change after interchanging jobs  $J_s$  and  $J_k$ . This means that the offset between  $M_1$  and  $M_2$  is kept in both cases, as it is depicted in Fig. 6 (cf.  $\pi_3$  on  $M_1$  and  $\pi_4$  on  $M_2$ ). Hence, the criterion value is influenced only by the change of idle times before jobs  $J_s$  and  $J_k$  and by the change of the schedule offsets after these jobs before  $\pi_3$  and  $\pi_4$ .

First, the schedule from time zero to the end of  $\pi_3$  will be considered. The difference between idle time before  $J_s$  in  $\pi$  and  $J_k$  in  $\pi'$  is equal to

$$(I_s - \hat{I}_k) = \max\{0, p_{1s} - \Delta\pi_1\} - \max\{0, p_{1k} - \Delta\pi_1\}. \quad (55)$$

The assumption that  $p_{1s} \geq p_{1k}$  implies that  $I_s \geq \hat{I}_k$  and  $(I_s - \hat{I}_k) \geq 0$ . This means that idle time before  $J_k$  in  $\pi'$  is not larger than before  $J_s$  in  $\pi$ . Moreover, the schedule offset before  $\pi_3$  changes by the value

$$\begin{aligned} \Delta\pi_s - \Delta\hat{\pi}_k &= (\max\{0, \Delta\pi_1 - p_{1s}\} + p_{2s}) \\ &\quad - (\max\{0, \Delta\pi_1 - p_{1k}\} + p_{2k}) \\ &= \max\{0, \Delta\pi_1 - p_{1s}\} \\ &\quad - \max\{0, \Delta\pi_1 - p_{1k}\} - (p_{2k} - p_{2s}). \end{aligned} \quad (56)$$

Since  $p_{1s} \geq p_{1k}$ , there is, in Eq. (56),  $\max\{0, \Delta\pi_1 - p_{1s}\} \leq \max\{0, \Delta\pi_1 - p_{1k}\} \Rightarrow \max\{0, \Delta\pi_1 - p_{1s}\} - \max\{0, \Delta\pi_1 - p_{1k}\} \leq 0$ . Because  $J_s$  succeeds  $J_k$  in Johnson's order, there is  $p_{2s} \leq p_{2k} \Rightarrow p_{2s} - p_{2k} \leq 0 \Rightarrow -(p_{2k} - p_{2s}) \leq 0$  and Eq. (56) reduces to  $\Delta\pi_s - \Delta\hat{\pi}_k \leq 0$ . This means that the schedule offset before  $\pi_3$  in  $\pi'$  is not smaller than in  $\pi$ . The possible increase of the schedule offset can reduce idle time at the beginning of  $\pi_3$  or within this subschedule (actually, the value  $-(\Delta\pi_s - \Delta\hat{\pi}_k)$  corresponds to the length of the artificial idle time before  $\pi_3$  on  $M_1$  in Fig. 6).

Now, let's consider the remaining part of the schedule from the end of  $\pi_3$  to the beginning of  $\pi_4$ . Since it is assumed that  $\pi_3$  is the same in  $\pi'$  as in  $\pi$ , the schedule offset before  $J_k$  in  $\pi$  and  $J_s$  in  $\pi'$  is identical and equal to  $\Delta\pi_3$ . The difference between idle times before  $J_k$  in  $\pi$  and  $J_s$  in  $\pi'$  is equal to

$$(I_k - \hat{I}_s) = \max\{0, p_{1k} - \Delta\pi_3\} - \max\{0, p_{1s} - \Delta\pi_3\}. \quad (57)$$

Because  $p_{1s} \geq p_{1k}$ , idle time before  $J_s$  in  $\pi'$  is not smaller than before  $J_k$  in  $\pi$  and  $(I_k - \hat{I}_s) \leq 0$ . Additionally, the schedule offset before  $\pi_4$  changes by the value

$$\begin{aligned} \Delta\pi_k - \Delta\hat{\pi}_s &= (\max\{0, \Delta\pi_3 - p_{1k}\} + p_{2k}) \\ &\quad - (\max\{0, \Delta\pi_3 - p_{1s}\} + p_{2s}) \\ &= \max\{0, \Delta\pi_3 - p_{1k}\} \\ &\quad - \max\{0, \Delta\pi_3 - p_{1s}\} - (p_{2s} - p_{2k}). \end{aligned} \quad (58)$$

The assumptions that  $p_{1s} \geq p_{1k}$  and  $p_{2s} \leq p_{2k}$  implies that  $\Delta\pi_k - \Delta\hat{\pi}_s \geq 0$ . This means that the offset before  $J_s$  in  $\pi'$  is not larger than before  $J_k$  in  $\pi$ . It may result in an additional idle time before  $\pi_4$  (cf. Fig. 6).

Summing up, before the schedule  $\pi_3$  the solution improves as a result of interchanging jobs  $J_s$  and  $J_k$ , and it deteriorates after  $\pi_3$ . The possible reduction of idle time at the beginning of the schedule ( $I_s - \hat{I}_k \geq 0$ ) has to be decreased by the additional idle time which appears at the end of it ( $I_k - \hat{I}_s \leq 0$ ). Moreover, shifting  $\pi_3$  to the left on  $M_1$  (cf. Fig. 6), no precedence constraint is violated. In consequence, the schedule offset before  $\pi_3$  increases by the value  $-(\Delta\pi_s - \Delta\hat{\pi}_k) \geq 0$ . A larger schedule offset can result only in a smaller idle time in the succeeding part of schedule  $\pi'$ , i.e. between  $J_k$  and  $\pi_3$ . However, this criterion improvement is reduced before  $\pi_4$  by the value  $-(\Delta\pi_k - \Delta\hat{\pi}_s) \leq 0$ . In consequence, the total change in idle time is equal to:

$$\begin{aligned} \Delta I &= (I_s - \hat{I}_k) + (I_k - \hat{I}_s) - (\Delta\pi_s - \Delta\hat{\pi}_k) \\ &\quad - (\Delta\pi_k - \Delta\hat{\pi}_s). \end{aligned} \quad (59)$$

Combining Eq. (59) with (55–58), one obtains

$$\begin{aligned} \Delta I &= (\max\{0, p_{1s} - \Delta\pi_1\} - \max\{0, p_{1k} - \Delta\pi_1\}) \\ &\quad + (\max\{0, p_{1k} - \Delta\pi_3\} - \max\{0, p_{1s} - \Delta\pi_3\}) \\ &\quad - (\max\{0, \Delta\pi_1 - p_{1s}\} - \max\{0, \Delta\pi_1 - p_{1k}\}) \\ &\quad - (p_{2k} - p_{2s}) \\ &\quad - (\max\{0, \Delta\pi_3 - p_{1k}\} - \max\{0, \Delta\pi_3 - p_{1s}\}) \\ &\quad - (p_{2s} - p_{2k}) \\ &= \max\{0, p_{1s} - \Delta\pi_1\} - \max\{0, \Delta\pi_1 - p_{1s}\} \\ &\quad + \max\{0, \Delta\pi_1 - p_{1k}\} - \max\{0, p_{1k} - \Delta\pi_1\} \\ &\quad + \max\{0, p_{1k} - \Delta\pi_3\} - \max\{0, \Delta\pi_3 - p_{1k}\} \\ &\quad + \max\{0, \Delta\pi_3 - p_{1s}\} - \max\{0, p_{1s} - \Delta\pi_3\} \\ &\quad + (p_{2k} - p_{2s}) + (p_{2s} - p_{2k}). \end{aligned} \quad (60)$$

Taking into account that for any integer  $x$

$$\max\{0, x\} - \max\{0, -x\} = x, \quad (61)$$

Equation (60) reduces to  $\Delta I = p_{1s} - \Delta\pi_1 + \Delta\pi_1 - p_{1k} + p_{1k} - (\pi_3 + (\pi_3 - p_{1s} + (p_{2k} - p_{2s}) - (p_{2k} - p_{2s})) = 0$ . The total idle time does not change after interchanging jobs  $J_s$  and  $J_k$ . Moreover, some idle time units are shifted to the right, possibly after a common due date, reducing in this way the total late work in the system. In consequence the schedule  $\pi'$  is not worse than  $\pi$ .

**Case 2.** Let's assume that  $J_k$  in  $\pi'$  starts earlier than  $J_s$  in  $\pi$ . This means that there exists a sequence  $\pi_2$  which is scheduled before  $J_s$  in  $\pi$  and after  $J_k$  in  $\pi'$ . Exemplary



solutions  $\pi$  and  $\pi'$  are depicted in Fig. 7. Some partial schedules  $\pi_1, \pi_3, \pi_4$  can be empty. Actually, the proof for Case 2 is a straightforward consequence of Case 1. In the analysis for Case 1, no assumption on the structure of a subschedule preceding  $J_s$  in  $\pi$  was formulated. Particularly, the assumption that all early jobs are sequenced in Johnson's order was not crucial for the correctness of the proof for Case 1.

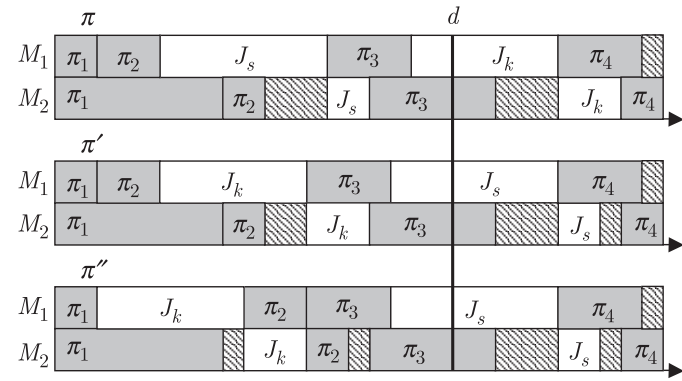


Fig. 7. Exemplary schedules  $\pi, \pi'$  and  $\pi''$ , where  $J_k \in J^2$  is interchanged with  $J_s \in J^2 (p_{1k} \leq p_{1s} \wedge p_{2s} \leq p_{2k})$  and there exists at least one job executed before  $J_s$  in  $\pi$  and after  $J_k$  in  $\pi'$

In consequence, interchanging jobs  $J_s$  and  $J_k$  in  $\pi$ , one obtain a new solution  $\pi'$  as depicted in Fig. 7, corresponding to Case 1 (where  $J_k$  and  $\pi_2$  are not in Johnson's order). Case 1 shows that the total idle time in  $\pi'$  is not larger than in  $\pi$ . Actually, in  $\pi'$  some idle time can be shifted to the right in comparison to  $\pi$  causing the possible decrease of the criterion value. In consequence  $\pi'$  is not worse than  $\pi$ .

Imposing Johnson's order on all early jobs causes shifting subschedule  $\pi_2$  after  $J_k$  in a final schedule  $\pi''$ . The starting times of  $\pi_3$  and the succeeding subschedules on  $M_1$  do not change after such a modification. Since Johnson's order is optimal from the schedule length point of view, the completion time of  $\pi_2$  can be only smaller in  $\pi''$  than the completion time of  $J_k$  in  $\pi'$ . This means that some idle time can be shifted to the right, possibly after a common due date, reducing the criterion value in this way.

Summing up,  $\pi''$  is not worse than  $\pi'$ , and, consequently, it is not worse than the original schedule  $\pi$ . The repeated analysis of the remaining pairs of jobs  $J_s, J_k \in J^2$  such that  $J_s \in E \wedge J_k \in L \wedge p_{1k} \leq p_{1s} \wedge p_{2k} \geq p_{2s}$ , proves Theorem 3.

The presented theorems suggest a strategy of constructing an optimal solution for problem  $F2|d_j = d|Y$ . If there is a pair of jobs  $J_k, J_s$  such that  $p_{1k} \leq p_{1s}$  and  $J_k$  precedes  $J_s$  in Johnson's order ( $J_k \rightarrow J_s$ ), and in a certain solution  $J_s$  is processed early, while  $J_k$  is late, then one can improve a current schedule by executing  $J_k$  before a common due date by:

- shifting  $J_k$  before  $J_s$ , if  $J_k \in J^1$  (for  $J_s \in J^1$  based on Theorem 1, for  $J_s \in J^2$  based on Theorem 2),
- interchanging  $J_k$  and  $J_s$ , if  $J_k \in J^2$  (it has to be  $J_s \in J^2$ , based on Theorem 3).

These two cases can be intuitively justified. The shorter processing time on the first machine ( $p_{1k} \leq p_{1s}$ ), denotes that the possible idle time before job  $J_k$  can be smaller than before job  $J_s$ . Moreover, processing a job from  $J^1$  early is usually profitable for the criterion value, because it does not decrease the offset between machines in a succeeding subschedule: the larger offset the smaller idle time before a succeeding job. On the contrary, jobs from  $J^2$  processed early may cause the offset decrease, since they have shorter second task than the first one. However, if  $J_k \in J^2$ , then the offset decrease for  $J_k$  is smaller than for  $J_s$ , because  $p_{1k} \leq p_{1s}$  and  $p_{2k} \geq p_{2s}$ , it makes interchanging these jobs profitable for the criterion value.

All the theorems presented in the paper concern pairs of jobs  $J_k$  and  $J_s$  such that  $p_{1k} \leq p_{1s}$  and  $J_k$  precedes  $J_s$  in Johnson's order ( $J_k \rightarrow J_s$ ). If  $J_k$  succeeds  $J_s$  in Johnson's order ( $J_s \rightarrow J_k$ ), then executing  $J_k$  early does not influence the schedule before  $J_s$  and one cannot estimate, whether moving  $J_k$  to the set of early jobs is profitable for the schedule quality or not with regard to  $J_s$ . On the other hand, interchanging  $J_s$  with  $J_k$  does not always result in the improvement of the criterion value, as it is shown in Fig. 8 (for  $J_k \in J^2$  and  $J_s \in J^1$ ) and in Fig. 9 (for  $J_k, J_s \in J^2$  and  $p_{2s} > p_{2k}$ ).

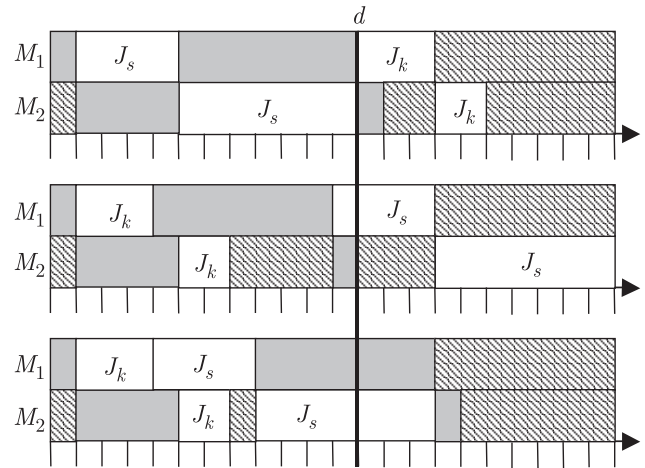


Fig. 8. Exemplary schedules with  $J_k \in J^2 \wedge J_s \in J^1 \wedge p_{1k} \leq p_{1s}$ , where executing  $J_k$  early instead of  $J_s$  is not profitable for the total late work value

These observations complete the analysis of all possible cases for jobs  $J_k$  and  $J_s$  with  $p_{1k} \leq p_{1s}$ , which are as follows:

- $J_k \rightarrow J_s \wedge J_k \in J^1 \wedge J_s \in J^1$ : shifting  $J_k$  before  $J_s$  is profitable, cf. Theorem 1,
- $J_k \rightarrow J_s \wedge J_k \in J^1 \wedge J_s \in J^2$ : shifting  $J_k$  before  $J_s$  is profitable, cf. Theorem 2,

- $J_k \rightarrow J_s \wedge J_k \in J^2 \wedge J_s \in J^1$ : the case is impossible,  $J^1$  precedes  $J^2$  in Johnson's order,
- $J_k \rightarrow J_s \wedge J_k \in J^2 \wedge J_s \in J^2$ : interchanging  $J_k$  and  $J_s$  is profitable, cf. Theorem 3,
- $J_s \rightarrow J_k \wedge J_k \in J^1 \wedge J_s \in J^1$ : the case is impossible,  $J_s \rightarrow J_k$  in Johnson's order means that  $p_{1s} \leq p_{1k}$  (if  $p_{1s} = p_{1k}$ , then Theorem 1 can be applied),
- $J_s \rightarrow J_k \wedge J_k \in J^1 \wedge J_s \in J^2$ : the case is impossible,  $J^1$  precedes  $J^2$  in Johnson's order,
- $J_s \rightarrow J_k \wedge J_k \in J^2 \wedge J_s \in J^1$ : processing  $J_k$  early might not be profitable, cf. Fig. 8,
- $J_s \rightarrow J_k \wedge J_k \in J^2 \wedge J_s \in J^2$ : processing  $J_k$  early might not be profitable, cf. Fig. 9.

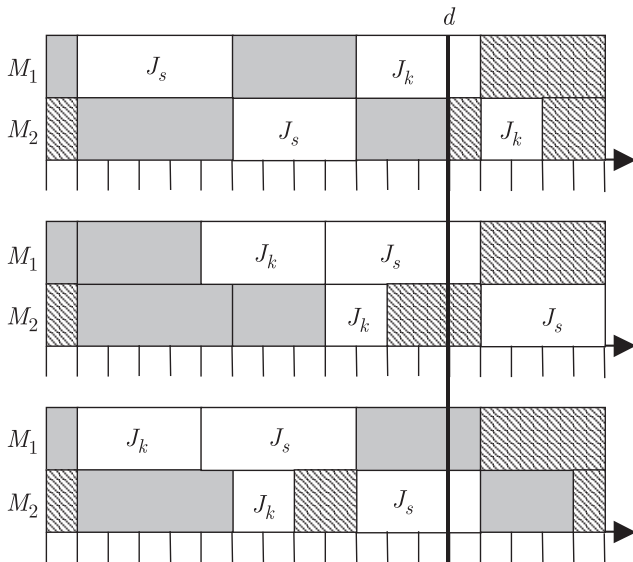


Fig. 9. Exemplary schedules with  $J_k, J_s \in J^2 \wedge p_{1k} \leq p_{1s} \wedge p_{2s} > p_{2k}$ , where executing  $J_k$  early instead of  $J_s$  is not profitable for the total late work value

#### 4. Conclusions

The paper presents three dominance relations for the two-machine flow shop problem with a common due date and the late work criterion,  $F2|d_j = d|Y$ , which describe the special features of an optimal solution of this scheduling case. They state that constructing an optimal sequence of jobs one should select early jobs based on their processing times on the first machine, preferring activities with a shorter first task.

These results make it possible to continue the research on problem  $F2|d_j = d|Y$  in two interesting directions. First, the efficiency of dominance relations will be checked in practice, by implementing a branch and bound approach. The presented theorems enable to truncate some branches in the search tree representing the solution process of the B&B method. Partial solutions, partial permutations, which are dominated in terms of the rules formulated in the paper can be discarded, usually reducing the run time of the exact approach.

On the other hand, the features of an optimal solution pointed out in the considered theorems will be important components of heuristic or metaheuristic approaches, whose efficiency will be validated mainly in the computational experiments. However, in the case of heuristic algorithms, the results given in the work might be also the basis for the theoretical analysis of their behavior in the worst case.

Thus, the results on  $F2|d_j = d|Y$  presented in the paper are the starting point for the further theoretical and computational studies on this scheduling problem.

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