BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 55, No. 1, 2007

LMI-based strategies for designing observers and unknown input observers for non-linear discrete-time systems

J. KORBICZ[1]*, M. WITCZAK[1], and V. PUIG[2]

[1]Institute of Control and Computation Engineering, University of Zielona Góra, 50 Podgórna, 65-246 Zielona Góra, Poland,
[2]Automatic Control Departament – Campus de Terrassa, Universidad Politécnica de Cataluña,
Rambla Sant Nebridi, 10, 08222 Terrassa, Spain

Abstract. The paper deals with the problems of designing observers and unknown input observers for discrete-time Lipschitz non-linear systems. In particular, with the use of the Lyapunov method, three different convergence criteria of the observer are developed. Based on the achieved results, three different design procedures are proposed. Then, it is shown how to extend the proposed approach to the systems with unknown inputs. The final part of the paper presents illustrative examples that confirm the effectiveness of the proposed techniques. The paper also presents a MATLAB® function that implements one of the design procedures.

Key words: non-linear systems, state estimation, observers, fault diagnosis, convergence, LMI.

1. Introduction

A continuous increase in the complexity, efficiency, and reliability of the modern industrial makes the problem Fault Detection and Isolation (FDI) [1] one of the most important research directions underlying contemporary automatic control. There is no doubt that the theory (and practice, as a consequence) of fault diagnosis and control is well-developed and mature for linear systems only [1,2]. There is also a number of different approaches that can be employed to settle the robustness problems regarding model uncertainty of linear systems [1,2]. Another kind of solutions that may increase the performance of the FDI scheme is based on an appropriate scheduling of the control test signals in such a way as to gain as much information as possible about the system being supervised [3]. There are also techniques that utilize multi-objective optimization [2,4] to improve the performance of FDI. Unfortunately, these techniques can be used for linear systems only.

Observers are commonly used in both control and fault diagnosis schemes of non-linear systems [1,2,5,6,7,8]. Undoubtedly, the most common approach is to use robust observers, such as the Unknown Input Observer (UIO) [2,9], which can tolerate a degree of model uncertainty and hence increase the reliability of fault diagnosis. Although the origins of UIOs can be traced back to the early 1970's (cf. the seminal work of Wang et al. [10]) the problem of designing such observers is still of paramount importance both from the theoretical and practical viewpoints. A large amount of knowledge on using these techniques for model-based fault diagnosis has been accumulated through the literature for the last three decades (see [1,2] and the references therein). Generally, the design problems regarding UIOs can be divided into the three distinct categories:

Design of UIOs for linear deterministic systems. Apart from the seminal paper of Wang et al. [10] it is worth to note a few pioneering and important works in this area, namely: the geometric approach by Bhattacharyya [11], the inversion algorithm by Kobayashi and Nakamizo [12], the algebraic approach by Hou and Müller [13] and finally the approach by Chen, Patton and Zhang [14]. The reader is also referred to the recently published developments, e.g. [15].

Design of UIOs for linear stochastic systems. Most design techniques concerning such a class of linear systems make use of the ideas for linear deterministic systems along with the Kalman filtering strategy. Thus, the resulting approaches can be perceived as Kalman filters for linear systems with unknown inputs. The representative approaches of this group were developed by: Chen, Patton and Zhang [2,14], Darouach and Zasadzinski [16], Hou and Patton [17], and finally Keller and Darouach [18].

A significantly different approach was proposed in [8]. Instead of using the Kalman filter-like approach, the author employed the bounded-error state estimation technique [19] but the way of decoupling the unknown input remained the same as that in [18].

Design of UIOs for non-linear systems. The design approaches developed for non-linear systems can generally be divided into three categories:

– **non-linear state-transformation-based techniques:** apart from a relatively large class of systems for which they can be applied, even if the non-linear

*e-mail: j.korbicz@issi.uz.zgora.pl

transformation is possible it leads to another non-linear system and hence the observer design problem remains open (see [5,9] and the references therein).

- **linearization-based techniques:** such approaches are based on a similar strategy as that for the extended Kalman filter [1]. In [7,8] the author proposed an extended unknown input observer for non-linear systems. He also proved that the proposed observer is convergent under certain nonrestrictive conditions.
- **observers for particular classes of non-linear systems:** for example UIOs for polynomial and bilinear systems [20,21] or UIOs for Lipschitz systems [22,23].

Taking into account the presented state-of-the-art regarding observers and unknown input observers for non-linear systems, the number of real world applications (not only simple simulated systems) of non-linear observers should proliferate. Unfortunately, this is not the case. The main reason of such a situation is related with a relatively high design complexity of non-linear observers [5,24]. This does not encourage engineers to apply them in an industrial reality. Indeed, apart from the theoretically large potential of the observer-based schemes, their computer implementation cause serious problems for engineers that are, usually, not fluent in a complex mathematical description involved in the theoretical developments.

Taking into account the above discussion, one objective of this paper is to propose a novel approach to designing observers for non-linear discrete-time systems. Another objective is to show how to extend the proposed technique to the systems with unknown input, i.e. to propose a design procedure of an UIO. The final objective of the paper is to propose a MATLAB®-based software that can efficiently be used for solving various practical problems involving application of observers. Apart from the purely commercial character of this numerical computation software, it is widely used in the control engineering community. This was the main reason for using it for the computer implementation of the proposed approach.

The paper is organized as follows. Section 2 presents an introductory background regarding observers and, in particular, observers for Lipschitz non-linear systems being the subject of the paper. Section 3 presents a comprehensive convergence analysis of the proposed observer with the use of the Lyapunov method. In Section 4, convenient and effective design procedures are proposed. In Section 5, a straightforward approach for extending the proposed techniques to the systems with unknown inputs is described and carefully discussed. Section 6 presents a number of illustrative examples that confirm the effectiveness of the proposed approaches. Finally, the MATLAB®-based computational procedure is presented in Appendix.

2. Preliminaries

A large amount of knowledge on designing observers for non-linear systems has been accumulated through the literature since the beginning of the 1970s (see, e.g., [6] and the references therein). A customary approach is to linearize the non-linear model around the current state estimate, and then to apply techniques for linear systems, as is the case for the extended Kalman filter (see, e.g., [25,26] and the references therein). Unfortunately, this strategy works well only when the linearization does not cause a large mismatch between the linear model and non-linear behaviour of the system. To improve the effectiveness of state estimation, it is necessary to restrict the class of non-linear systems while designing observers. For example, the Lipschitz systems that can be described as follows:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{h}(\boldsymbol{y}(t), \boldsymbol{u}(t)) + \boldsymbol{g}\left(\boldsymbol{x}(t), \boldsymbol{u}(t)\right), \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t), \tag{2}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ stands for the state vector, $\boldsymbol{y}(t) \in \mathbb{R}^m$ is the output, $\boldsymbol{u}(t) \in \mathbb{R}^r$ is the input, $\boldsymbol{h}(\boldsymbol{y}(t), \boldsymbol{u}(t))$ and $\boldsymbol{g}\left(\boldsymbol{x}(t), \boldsymbol{u}(t)\right)$ are a non-linear functions, where $\boldsymbol{g}\left(\boldsymbol{x}(t), \boldsymbol{u}(t)\right)$ satisfies:

$$\|\boldsymbol{g}\left(\boldsymbol{x}_1, \boldsymbol{u}\right) - \boldsymbol{g}\left(\boldsymbol{x}_2, \boldsymbol{u}\right)\|_2 \leq \gamma \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2, \ \forall \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u} \tag{3}$$

where $\gamma > 0$ stands for the Lipschitz constant. Many non-linear systems can be described by (1), e.g. sinusoidal non-linearities satisfy (3), even polynomial non-linearities satisfy (3) assuming that $\boldsymbol{x}(t)$ is bounded. This means that (1)–(2) can be used for describing a wide class of non-linear systems, which is very important from the point of view of potential industrial applications.

The first solution for the state estimation of (1)–(2) was developed by Thau [27]. Assuming that the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable, Thau proposed a convergence condition but he did not provide an effective design procedure of the observer. In other words, in light of this approach, the observer has to be designed with a trial-and-error procedure that amounts to solving a large number of Lyapunov equations and then checking the convergence conditions. Many different authors followed the similar procedure but they proposed less restrictive convergence conditions (see, e.g. [28]). Finally, [29–31] proposed a more effective observer design. In particular, in [30] the authors employed the concept of the distance to unobservability of the pair $(\boldsymbol{A}, \boldsymbol{C})$ and proposed an iterative coordinate transformation technique reducing the Lipschitz constant. In [29] the authors employed and improved the results of [30] but the proposed design procedure does not seem straightforward. In [31] the author reduced the observer design problem to a global optimization one. The main disadvantage of this approach is that the proposed algorithm does not guarantee to obtain a global optimum. Thus, many trial-and-error steps have to be carried out to obtain a satisfactory solution. Recently, in [23] the authors proposed the so-called dynamic observer with a mixed binary search and $\mathcal{H}_\infty$ optimization procedure.

Unfortunately, the theory and practice concerning observers for discrete-time counterpart of (1)–(2):

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{h}(\boldsymbol{y}_k, \boldsymbol{u}_k) + \boldsymbol{g}\left(\boldsymbol{x}_k, \boldsymbol{u}_k\right), \quad (4)$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{C}\boldsymbol{x}_{k+1}, \quad (5)$$

are significantly less mature than these for (1)–(2). Indeed, there are a few papers only [32,33] dealing with discrete-time observers. The authors of the above works propose different parameterizations of the observer but the common disadvantage of these approaches is that a trial-and-error procedure has to be employed that boils down to solving a large number of Lyapunov equations. Moreover, the authors do not provide convergence conditions similar to those for the continuous-time observers [27,28].

3. Convergence analysis

Let us consider an observer for the system (4)–(5) described by the following equation:

$$\hat{\boldsymbol{x}}_{k+1} = \boldsymbol{A}\hat{\boldsymbol{x}}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{h}(\boldsymbol{y}_k, \boldsymbol{u}_k) + \boldsymbol{g}\left(\hat{\boldsymbol{x}}_k, \boldsymbol{u}_k\right) + \boldsymbol{K}(\boldsymbol{y}_k - \boldsymbol{C}\hat{\boldsymbol{x}}_k), \quad (6)$$

where $\boldsymbol{K}$ stands for the gain matrix. The subsequent part of this section shows three theorems that present three different convergence conditions of (6). Following Thau [27] and other researchers, let us assume that the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable. Let $\underline{\sigma}(\cdot)$ and $\bar{\sigma}(\cdot)$ stand for the minimum and maximum singular values, and $\boldsymbol{P} = \boldsymbol{P}^T$, $\boldsymbol{P} > \boldsymbol{0}$ be a solution of the following Lyapunov equation:

$$\boldsymbol{Q} = \boldsymbol{P} - \boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{A}_0, \quad \boldsymbol{A}_0 = \boldsymbol{A} - \boldsymbol{K}\boldsymbol{C}, \quad (7)$$

where $\boldsymbol{A}_0$ is a stable matrix, and $\boldsymbol{Q} = \boldsymbol{Q}^T$, $\boldsymbol{Q} > 0$.

Theorem 1. Let us consider an observer (6) for the systems described by (4)–(5). If the Lipschitz constant γ (cf. (3)) satisfies:

$$\gamma < \sqrt{\frac{\underline{\sigma}\left(\boldsymbol{Q} - \frac{1}{2}\boldsymbol{P}\right)}{\bar{\sigma}\left(\boldsymbol{P}\right)}}, \quad \boldsymbol{Q} - \frac{1}{2}\boldsymbol{P} \succ \boldsymbol{0} \quad (8)$$

then the observer (6) is asymptotically convergent.

Proof. Let us define the state estimation error for (6):

$$\boldsymbol{e}_k = \boldsymbol{x}_k - \hat{\boldsymbol{x}}_k, \quad (9)$$

and

$$\boldsymbol{z}_k = \boldsymbol{g}\left(\boldsymbol{x}_k, \boldsymbol{u}_k\right) - \boldsymbol{g}\left(\hat{\boldsymbol{x}}_k, \boldsymbol{u}_k\right). \quad (10)$$

Substituting (4)–(5), (6) and (10) into (9) gives:

$$\boldsymbol{e}_{k+1} = \boldsymbol{A}_0\boldsymbol{e}_k + \boldsymbol{z}_k. \quad (11)$$

Let us define the following Lyapunov function:

$$V_{k+1} = \boldsymbol{e}_{k+1}^T\boldsymbol{P}\boldsymbol{e}_{k+1}, \quad (12)$$

and then inserting (11) one can get:

$$V_{k+1} = \boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{A}_0\boldsymbol{e}_k + 2\boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k + \boldsymbol{z}_k^T\boldsymbol{P}\boldsymbol{z}_k. \quad (13)$$

According to the Lyapunov theorem, the observer (6) is asymptotically convergent iff:

$$\Delta V = V_{k+1} - V_k < 0. \quad (14)$$

Substituting (12) and (13) into (14) yields:

$$\Delta V = \boldsymbol{e}_k^T\left[\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{A}_0 - \boldsymbol{P}\right]\boldsymbol{e}_k + 2\boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k + \boldsymbol{z}_k^T\boldsymbol{P}\boldsymbol{z}_k < 0. \quad (15)$$

Knowing that:

$$\left(\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{A}_0\boldsymbol{z}_k - \boldsymbol{P}^{\frac{1}{2}}\boldsymbol{z}_k\right)^T\left(\boldsymbol{P}^{\frac{1}{2}}\boldsymbol{A}_0\boldsymbol{z}_k - \boldsymbol{P}^{\frac{1}{2}}\boldsymbol{z}_k\right) \geq 0,$$

one can obtain that:

$$2\boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k \leq \boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{A}_0\boldsymbol{e}_k + \boldsymbol{z}_k^T\boldsymbol{P}\boldsymbol{z}_k. \quad (16)$$

Inserting (16) into (15) yields:

$$\Delta V \leq 2\boldsymbol{e}_k^T\left[\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{A}_0 - \frac{1}{2}\boldsymbol{P}\right]\boldsymbol{e}_k + 2\boldsymbol{z}_k^T\boldsymbol{P}\boldsymbol{z}_k < 0. \quad (17)$$

Using (3) it can be shown that:

$$\boldsymbol{z}_k^T\boldsymbol{P}\boldsymbol{z}_k \leq \gamma^2\bar{\sigma}\left(P\right)\boldsymbol{e}_k^T\boldsymbol{e}_k. \quad (18)$$

Substituting (18) into (17) gives:

$$\Delta V \leq 2\boldsymbol{e}_k^T\left[\gamma^2\bar{\sigma}\left(P\right)\boldsymbol{I} - \left[\boldsymbol{Q} - \frac{1}{2}\boldsymbol{P}\right]\right]\boldsymbol{e}_k < 0. \quad (19)$$

The condition (19) is equivalent to:

$$\gamma < \sqrt{\frac{1}{\bar{\sigma}\left(\boldsymbol{P}\right)}\frac{\boldsymbol{e}_k^T\left[\boldsymbol{Q} - \frac{1}{2}\boldsymbol{P}\right]\boldsymbol{e}_k}{\boldsymbol{e}_k^T\boldsymbol{e}_k}}. \quad (20)$$

Using the bound of the Rayleigh quotient, i.e. $\boldsymbol{e}_k^T\left[\boldsymbol{Q} - \frac{1}{2}\boldsymbol{P}\right]\boldsymbol{e}_k / \boldsymbol{e}_k^T\boldsymbol{e}_k \geq \underline{\sigma}\left(Q - \frac{1}{2}\boldsymbol{P}\right)$, it is possible to obtain (8), which completes the proof.

Theorem 2. Let us consider an observer (6) for the systems described by (4)–(5). If the Lipschitz constant γ (cf. (3)) satisfies:

$$\gamma < \sqrt{\frac{\underline{\sigma}\left(\boldsymbol{Q} - \boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{P}\boldsymbol{A}_0\right)}{\bar{\sigma}\left(\boldsymbol{P}\right) + 1}}, \quad \boldsymbol{Q} - \boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{P}\boldsymbol{A}_0 \succ \boldsymbol{0} \quad (21)$$

then the observer (6) is asymptotically convergent.

Proof. Using (3) and the Cauchy-Schwartz inequality, it can be shown that:

$$2\boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k \leq 2\gamma\|\boldsymbol{P}\boldsymbol{A}_0\boldsymbol{e}_k\|_2\|\boldsymbol{e}_k\|_2. \quad (22)$$

Applying the identity:

$$\left(\|\boldsymbol{P}\boldsymbol{A}_0\boldsymbol{e}_k\|_2 - \gamma\|\boldsymbol{e}_k\|_2\right)^2 \geq 0,$$

to (22) yields:

$$2\boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k \leq \boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{P}\boldsymbol{A}_0\boldsymbol{e}_k + \gamma^2\boldsymbol{e}_k^T\boldsymbol{e}_k. \quad (23)$$

Substituting (23) into (15) and then applying (18) leads to:

$$\Delta V \leq \boldsymbol{e}_k^T\left[\gamma^2(\bar{\sigma}\left(P\right) + 1)\boldsymbol{I} - \left[\boldsymbol{Q} - \boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{P}\boldsymbol{A}_0\right]\right]\boldsymbol{e}_k < 0. \quad (24)$$

Finally, it is straightforward to show that (24) is equivalent to (21), which completes the proof.

THEOREM 3. Let us consider an observer (6) for the systems described by (4)–(5). If the Lipschitz constant γ (cf. (3)) satisfies:

$$\gamma < \frac{\underline{\sigma}\left(\boldsymbol{Q}^{\frac{1}{2}}\right)}{\sqrt{\bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right)^2 + \bar{\sigma}\left(\boldsymbol{P}\right)} + \bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right)} \quad (25)$$

then the observer (6) is asymptotically convergent.

Proof. Using (15), (7) and (18) it can be shown that the convergence condition is:

$$\Delta V \leq \boldsymbol{e}_k^T\left[\gamma^2\bar{\sigma}\left(\boldsymbol{P}\right)\boldsymbol{I} - \boldsymbol{Q}\right]\boldsymbol{e}_k + 2\boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k < 0. \quad (26)$$

and hence:

$$2\boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k < \boldsymbol{e}_k^T\left[\boldsymbol{Q} - \gamma^2\bar{\sigma}\left(\boldsymbol{P}\right)\boldsymbol{I}\right]\boldsymbol{e}_k,$$

which is equivalent to:

$$2\boldsymbol{z}_k^T\boldsymbol{P}\boldsymbol{A}_0\boldsymbol{e}_k < \boldsymbol{e}_k^T\left[\boldsymbol{Q} - \gamma^2\bar{\sigma}\left(\boldsymbol{P}\right)\boldsymbol{I}\right]\boldsymbol{e}_k. \quad (27)$$

Inequality (27) can be written as follows (cf. [34]):

$$2\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k\right)^T\left(\boldsymbol{Q}^{\frac{1}{2}}\boldsymbol{e}_k\right) < \left(\boldsymbol{Q}^{\frac{1}{2}}\boldsymbol{e}_k\right)^T\left(\boldsymbol{Q}^{\frac{1}{2}}\boldsymbol{e}_k\right) - \gamma^2\bar{\sigma}\left(\boldsymbol{P}\right)\boldsymbol{e}_k^T\boldsymbol{e}_k,$$

and, hence, the convergence condition is:

$$2\left\|\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k\right\|_2 < \left\|\boldsymbol{Q}^{\frac{1}{2}}\boldsymbol{e}_k\right\|_2 - \gamma^2\bar{\sigma}\left(\boldsymbol{P}\right)\frac{\|\boldsymbol{e}_k\|_2^2}{\left\|\boldsymbol{Q}^{\frac{1}{2}}\boldsymbol{e}_k\right\|_2}. \quad (28)$$

Using (18), it can be shown that:

$$\left\|\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k\right\|_2 \leq \gamma\bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right)\|\boldsymbol{e}_k\|_2, \quad (29)$$

then knowing that:

$$\left\|\boldsymbol{Q}^{\frac{1}{2}}\boldsymbol{e}_k\right\|_2 \geq \underline{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\right)\|\boldsymbol{e}_k\|_2,$$

and

$$\frac{\|\boldsymbol{e}_k\|_2}{\left\|\boldsymbol{Q}^{\frac{1}{2}}\boldsymbol{e}_k\right\|_2} \leq \frac{1}{\underline{\sigma}\left(\boldsymbol{Q}^{\frac{1}{2}}\right)},$$

inequality (28) can be written as follows:

$$\frac{\bar{\sigma}\left(\boldsymbol{P}\right)}{\underline{\sigma}\left(\boldsymbol{Q}^{\frac{1}{2}}\right)}\gamma^2 + 2\gamma\bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right) - \underline{\sigma}\left(\boldsymbol{Q}^{\frac{1}{2}}\right) < 0. \quad (30)$$

Since (30) contains a quadratic function then it is clear that:

$$\gamma < \left(\sqrt{\bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right)^2 + \bar{\sigma}\left(\boldsymbol{P}\right)} + -\bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right)\right)\frac{\underline{\sigma}\left(\boldsymbol{Q}^{\frac{1}{2}}\right)}{\bar{\sigma}\left(\boldsymbol{P}\right)}. \quad (31)$$

Finally, using the identity:

$$\left(\sqrt{\bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right)^2 + \bar{\sigma}\left(\boldsymbol{P}\right)} - \bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right)\right) \cdot \left(\sqrt{\bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right)^2 + \bar{\sigma}\left(\boldsymbol{P}\right)} + \bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right)\right) = \bar{\sigma}\left(\boldsymbol{P}\right),$$

inequality (31) can be transformed into (25), which completes the proof.

Remark 1. The convergence criteria described by the above theorems are obtained by eliminating the term:

$$2\boldsymbol{e}_k^T\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{z}_k,$$

from (15) in three distinct ways. This means that the obtained criteria are relatively conservative and the scale of this conservatism is strongly related with the inaccuracy of a given elimination technique.

Remark 2. There is no doubt that there are particular choices of $\boldsymbol{Q}$ which will bring forth the least conservative bounds (8), (21) and (25), respectively. Unfortunately, the structural relation between $\boldsymbol{P}$ and $\boldsymbol{Q}$ of (7) cannot be resolved without first solving the Lyapunov equation. This is the main reason why it is impossible to chose one criterion that gives the least conservative bound of γ for an arbitrary matrix $\boldsymbol{Q}$.

Remark 3. Unfortunately, (8), (21) and (25) may merely serve as methods for checking the convergence but the gain matrix $\boldsymbol{K}$ has to be determined beforehand. This means that the design procedure boils down to selecting various gain matrices $\boldsymbol{K}$, solving Lyapunov equation (7), and then checking the convergence conditions (8), (21) and (25). There is no doubt that this is an ineffective and inconvenient approach.

Taking into account the above remarks, the objective of the subsequent section is to develop three different design procedure that are based on (8), (21) and (25).

4. Design procedures

4.1. Design procedure 1. It can easily be shown that (19) is equivalent to:

$$\gamma^2\bar{\sigma}\left(\boldsymbol{P}\right)\boldsymbol{I} + \boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{A}_0 - \frac{1}{2}\boldsymbol{P} \prec \boldsymbol{0}. \quad (32)$$

Assuming that $\bar{\sigma}\left(\boldsymbol{P}\right) < \beta$, $\beta > 0$, and knowing that $\bar{\sigma}\left(\boldsymbol{P}\right) < \beta$ is equivalent to $\beta - \beta^{-1}\boldsymbol{P}\boldsymbol{P} \succ \boldsymbol{0}$ which can be written in the following LMI form:

$$\begin{bmatrix}\beta\boldsymbol{I} & \boldsymbol{P}\\ \boldsymbol{P} & \beta\boldsymbol{I}\end{bmatrix} \succ \boldsymbol{0}, \quad \beta > 0, \quad \boldsymbol{P} \succ \boldsymbol{0}, \quad (33)$$

(32) can be transformed into a set of inequalities:

$$\gamma^2\beta\boldsymbol{I} + \boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{A}_0 - \frac{1}{2}\boldsymbol{P} \prec \boldsymbol{0}, \quad (34)$$

and (33). Inequality (34) can be written in the following form:

$$\begin{bmatrix}\frac{1}{2}\boldsymbol{P} - \gamma^2\beta\boldsymbol{I} & \boldsymbol{A}_0^T\\ \boldsymbol{A}_0 & \boldsymbol{P}^{-1}\end{bmatrix} \succ \boldsymbol{0}, \quad (35)$$

which is equivalent to:

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\boldsymbol{P} - \gamma^2\beta\boldsymbol{I} & \boldsymbol{A}_0^T \\ \boldsymbol{A}_0 & \boldsymbol{P}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{P} \end{bmatrix} \succ \boldsymbol{0}. \quad (36)$$

Finally, (36) can be written in the following form:

$$\begin{bmatrix} \frac{1}{2}\boldsymbol{P} - \gamma^2\beta\boldsymbol{I} & \boldsymbol{A}_0^T\boldsymbol{P} \\ \boldsymbol{P}\boldsymbol{A}_0 & \boldsymbol{P} \end{bmatrix} \succ \boldsymbol{0}. \quad (37)$$

Substituting $\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{L}$ into (37) yields the following LMI:

$$\begin{bmatrix} \frac{1}{2}\boldsymbol{P} - \gamma^2\beta\boldsymbol{I} & \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{C}^T\boldsymbol{L}^T \\ \boldsymbol{P}\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C} & \boldsymbol{P} \end{bmatrix} \succ \boldsymbol{0}. \quad (38)$$

Thus, the design procedure can be summarized as follows:

Step 1. Obtain γ for (4)–(5).
Step 2. Solve a set of LMIs: (33) and (38).
Step 3. Obtain the gain matrix $\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{L}$.

In spite of the simplicity and effectiveness of the proposed approach it cannot directly be applied to determine $\boldsymbol{K}$ maximizing γ for which the observer (6) is convergent. The objective of the subsequent part of this section is to tackle the-above-defined task. It can be observed that (38) can be transformed into the following form:

$$\begin{bmatrix} -\frac{1}{2}\boldsymbol{P} & \boldsymbol{C}^T\boldsymbol{L}^T - \boldsymbol{A}^T\boldsymbol{P} \\ \boldsymbol{L}\boldsymbol{C} - \boldsymbol{P}\boldsymbol{A} & -\boldsymbol{P} \end{bmatrix} \prec \lambda \begin{bmatrix} \beta\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \quad (39)$$

where $\lambda = -\gamma^2$. Thus, the task can be reduced to the generalized eigenvalue minimization problem [35,36] that can be formulated as follows:

$$\min_{\boldsymbol{P},\boldsymbol{L},\beta} \lambda$$

under the LMI constraints (33) and (39). As can be observed, the right hand side of (39) is semi-positive definite. The positivity of the right hand side of (39) is, usually, required for the well-posedness of the task and the applicability of the polynomial-time interior point methods [36]. For a simple remedy to this problem the reader is referred to [36].
It should be also strongly underlined that when the optimization problem described by Steps 1–3 (or in the form of the generalized eigenvalue minimization problem) cannot be solved due to its infeasibility then the only way out is to transform the original description of the system into an equivalent one with a smaller Lipschitz constant. Some guidance regarding such a strategy are given in [29,30]. Thus, due the the observability assumption, the algorithm is guaranteed to converge as $\gamma \to 0$. This is, of course, the common drawback of the existing approaches to the design of observers for Lipschitz non-linear systems (cf. [23,29,30]).

4.2. Design procedure 2. It can easily be shown that (24) is equivalent to:

$$\gamma^2(\bar{\sigma}(\boldsymbol{P}) + 1)\boldsymbol{I} + \boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{P}\boldsymbol{A}_0 - \boldsymbol{P} \prec \boldsymbol{0}. \quad (40)$$

Assuming that $\bar{\sigma}(\boldsymbol{P}) < \beta$, $\beta > 0$, and $\boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{P}\boldsymbol{A}_0 \prec \boldsymbol{X}$, $\boldsymbol{X} = \boldsymbol{X}^T$, which can be expressed as:

$$\begin{bmatrix} \boldsymbol{X} & \boldsymbol{A}_0^T\boldsymbol{P} \\ \boldsymbol{P}\boldsymbol{A}_0 & \boldsymbol{I} \end{bmatrix} \succ \boldsymbol{0}, \quad (41)$$

inequality (40) can be written as follows:

$$\begin{bmatrix} \boldsymbol{P} - \gamma^2(\beta+1)\boldsymbol{I} - \boldsymbol{X} & \boldsymbol{A}_0^T\boldsymbol{P} \\ \boldsymbol{P}\boldsymbol{A}_0 & \boldsymbol{P} \end{bmatrix} \succ \boldsymbol{0}. \quad (42)$$

Substituting $\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{L}$ into (41) and (42) yields the following set of LMIs:

$$\begin{bmatrix} \boldsymbol{X} & \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{C}^T\boldsymbol{L}^T \\ \boldsymbol{P}\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C} & \boldsymbol{I} \end{bmatrix} \succ \boldsymbol{0}, \quad (43)$$

and

$$\begin{bmatrix} \boldsymbol{P} - \gamma^2(\beta+1)\boldsymbol{I} - \boldsymbol{X} & \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{C}^T\boldsymbol{L}^T \\ \boldsymbol{P}\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C} & \boldsymbol{P} \end{bmatrix} \succ \boldsymbol{0}. \quad (44)$$

Thus, the new design procedure can be summarized as follows:

Step 1. Obtain γ for (4)–(5).
Step 2. Solve a set of LMIs: (33), (43), and (44).
Step 3. Obtain the gain matrix $\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{L}$.

Similarly as in Section 4.1, the selection of $\boldsymbol{K}$ maximizing γ for which the observer (6) is convergent can be formulated as the generalized eigenvalue minimization problem:

$$\min_{\boldsymbol{P},\boldsymbol{L},\boldsymbol{X},\beta} \lambda$$

under the LMI constraints (33), (43), and

$$\begin{bmatrix} \boldsymbol{X} - \boldsymbol{P} & \boldsymbol{C}^T\boldsymbol{L}^T - \boldsymbol{A}^T\boldsymbol{P} \\ \boldsymbol{L}\boldsymbol{C} - \boldsymbol{P}\boldsymbol{A} & -\boldsymbol{P} \end{bmatrix} \prec \lambda \begin{bmatrix} (\beta+1)\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}. \quad (45)$$

where (45) is obtained by suitably rearranging (44), and $\lambda = -\gamma^2$.

4.3. Design procedure 3. Inequality (30) can be transformed into an equivalent form:

$$\bar{\sigma}(\boldsymbol{P})\gamma^2 + 2\gamma\underline{\sigma}\left(\boldsymbol{Q}^{\frac{1}{2}}\right)\bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right) - \underline{\sigma}(\boldsymbol{Q}) < 0. \quad (46)$$

Knowing that:

$$\underline{\sigma}\left(\boldsymbol{Q}^{\frac{1}{2}}\right)\bar{\sigma}\left(\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{A}_0^T\boldsymbol{P}\right) \le \bar{\sigma}\left(\boldsymbol{A}_0^T\boldsymbol{P}\right), \quad (47)$$

inequality (46) can be written as:

$$\bar{\sigma}(\boldsymbol{P})\gamma^2 + 2\gamma\bar{\sigma}\left(\boldsymbol{A}_0^T\boldsymbol{P}\right) - \underline{\sigma}(\boldsymbol{Q}) < 0. \quad (48)$$

Assuming that $\bar{\sigma}(\boldsymbol{P}) < \beta$, $\beta > 0$, and $\bar{\sigma}\left(\boldsymbol{A}_0^T\boldsymbol{P}\right) < \delta$, $\delta > 0$, which can be expressed as:

$$\begin{bmatrix} \delta & \boldsymbol{A}_0^T\boldsymbol{P} \\ \boldsymbol{P}\boldsymbol{A}_0 & \delta \end{bmatrix} \succ \boldsymbol{0}, \quad \delta > 0. \quad (49)$$

Now it is straightforward to show that (48) can be represented by:

$$\boldsymbol{P} - \boldsymbol{A}_0^T\boldsymbol{P}\boldsymbol{A}_0 - \gamma^2\beta\boldsymbol{I} - 2\gamma\delta\boldsymbol{I} \succ \boldsymbol{0}. \quad (50)$$

Thus, inequality (50) can be written as follows:

$$\begin{bmatrix} \boldsymbol{P} - \gamma^2\beta\boldsymbol{I} - 2\gamma\delta\boldsymbol{I} & \boldsymbol{A}_0^T\boldsymbol{P} \\ \boldsymbol{P}\boldsymbol{A}_0 & \boldsymbol{P} \end{bmatrix} \succ \boldsymbol{0}. \quad (51)$$

Substituting $\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{L}$ into (49) and (51) yields the following set of LMIs:

$$\begin{bmatrix} \delta & \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{C}^T\boldsymbol{L}^T \\ \boldsymbol{PA} - \boldsymbol{LC} & \delta \end{bmatrix} \succ \boldsymbol{0}, \quad \delta > 0 \tag{52}$$

and

$$\begin{bmatrix} \boldsymbol{P} - \gamma^2\beta\boldsymbol{I} - 2\gamma\delta\boldsymbol{I} & \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{C}^T\boldsymbol{L}^T \\ \boldsymbol{PA} - \boldsymbol{LC} & \boldsymbol{P} \end{bmatrix} \succ \boldsymbol{0}. \tag{53}$$

Thus, the third design procedure can be summarized as follows:

Step 1. Obtain γ for (4)–(5).
Step 2. Solve a set of LMIs: (33), (52), and (53).
Step 3. Obtain the gain matrix $\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{L}$.

Similarly as in Section 4.1, the selection of $\boldsymbol{K}$ maximizing γ for which the observer (6) is convergent can be formulated as the generalized eigenvalue minimization problem. First, let us assume that:

$$-\boldsymbol{X} \prec \lambda\beta\boldsymbol{I}, \quad \boldsymbol{X} \succ \boldsymbol{0} \tag{54}$$

where $\boldsymbol{X} = \boldsymbol{X}^T$, $\lambda = -\gamma$. Thus, inequality (53) can be expressed as:

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{C}^T\boldsymbol{L}^T - \boldsymbol{A}^T\boldsymbol{P} \\ \boldsymbol{LC} - \boldsymbol{PA} & -\boldsymbol{P} \end{bmatrix} \prec \lambda \begin{bmatrix} \boldsymbol{X} + 2\delta\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}. \tag{55}$$

Finally, the generalized eigenvalue minimization problem boils down to: $\min\limits_{\boldsymbol{P},\boldsymbol{L},\boldsymbol{X},\beta,\delta} \lambda$, under the LMI constraints (33), (52), (54)-(55).

5. Design of an unknown input observer

Irrespective of the identification method selected for designing the model, there always exists the problem of model uncertainty, i.e. the model-reality mismatch. To overcome this problem, many approaches have been proposed [1,2]. Undoubtedly, the most common approach is to use robust observers, such as the unknown input observer [1,2,8], which can tolerate a degree of model uncertainty and hence increase the reliability of fault diagnosis. In such an approach, the model-reality mismatch can be represented by the so called unknown input. There are relatively scare works on designing UIOs for non-linear Lipschitz systems [22,23]. All the presented works deal with the continuous-time system.

Thus, the purpose of the subsequent part of this section is to present a straightforward approach for extending the techniques proposed in the preceding sections to the discrete-time Lipschitz systems with unknown inputs, which can be described as follows:

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{h}(\boldsymbol{y}_k, \boldsymbol{u}_k) + \boldsymbol{g}(\boldsymbol{x}_k, \boldsymbol{u}_k) + \boldsymbol{E}\boldsymbol{d}_k, \tag{56}$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{C}\boldsymbol{x}_{k+1}, \tag{57}$$

where $\boldsymbol{d}_k \in \mathbb{R}^q$, $q \leq m$ stands for an unknown input and $\boldsymbol{E}$ denotes its distribution matrix.

In order to use the techniques described in the preceding sections for the state estimation of the system (56)–(57) it is necessary to introduce some modifications concerning the unknown input. In the existing approaches, the unknown input is usually treated in two different ways. The first one (see e.g. [2]) relies on introducing an additional matrix into the state estimation equation, which is then used for de-coupling the effect of the unknown input on the state estimation error (and consequently on the residual signal). In the second one (see e.g. [18]), the system with an unknown input is suitably transformed into a system without it. In both cases the necessary condition for the existence of a solution to the unknown input de-coupling problem is:

$$\operatorname{rank}(\boldsymbol{CE}) = \operatorname{rank}(\boldsymbol{E}), \tag{58}$$

(see [2] for a comprehensive explanation). If condition (58) is satisfied, then it is possible to calculate $\boldsymbol{H} = (\boldsymbol{CE})^+ = \left[(\boldsymbol{CE})^T\boldsymbol{CE}\right]^{-1}(\boldsymbol{CE})^T$, where $(\cdot)^+$ stands for the pseudo-inverse of its argument. Thus, let us use the first of the above mentioned techniques for designing UIOs [18]. By multiplying (57) by $\boldsymbol{H}$ and then inserting (56) it is straightforward to show that:

$$\boldsymbol{d}_k = \boldsymbol{H}\left[\boldsymbol{y}_{k+1} - \boldsymbol{C}\left[\boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{h}(\boldsymbol{y}_k, \boldsymbol{u}_k) + \boldsymbol{g}(\boldsymbol{x}_k, \boldsymbol{u}_k)\right]\right]. \tag{59}$$

Substituting (59) into (56) gives:

$$\boldsymbol{x}_{k+1} = \bar{\boldsymbol{A}}\boldsymbol{x}_k + \bar{\boldsymbol{B}}\boldsymbol{u}_k + \bar{\boldsymbol{h}}(\boldsymbol{u}_k, \boldsymbol{y}_k) + \bar{\boldsymbol{g}}(\boldsymbol{x}_k, \boldsymbol{u}_k) + \bar{\boldsymbol{E}}\boldsymbol{y}_{k+1}, \tag{60}$$

where

$$\bar{\boldsymbol{A}} = \bar{\boldsymbol{G}}\boldsymbol{A},\ \bar{\boldsymbol{B}} = \bar{\boldsymbol{G}}\boldsymbol{B},\ \bar{\boldsymbol{g}}(\cdot) = \bar{\boldsymbol{G}}\boldsymbol{g}(\cdot),\ \bar{\boldsymbol{h}}(\cdot) = \bar{\boldsymbol{G}}\boldsymbol{h}(\cdot)$$
$$\bar{\boldsymbol{G}} = \boldsymbol{I} - \boldsymbol{EHC},\ \bar{\boldsymbol{E}} = \boldsymbol{EH}.$$

Thus, the unknown input observer for (56)–(57) is given as follows:

$$\hat{\boldsymbol{x}}_{k+1} = \bar{\boldsymbol{A}}\hat{\boldsymbol{x}}_k + \bar{\boldsymbol{B}}\boldsymbol{u}_k + \bar{\boldsymbol{h}}(\boldsymbol{u}_k, \boldsymbol{y}_k) + \bar{\boldsymbol{g}}(\boldsymbol{x}_k, \boldsymbol{u}_k) + \bar{\boldsymbol{E}}\boldsymbol{y}_{k+1} + \boldsymbol{K}(\boldsymbol{y}_k - \boldsymbol{C}\hat{\boldsymbol{x}}_k). \tag{61}$$

Now, let us consider the second of the above-mentioned approaches that can be used for designing the UIO [2]. For the sake of notational simplicity, let us start with the UIO for linear discrete-time systems:

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{E}\boldsymbol{d}_k,$$
$$\boldsymbol{y}_{k+1} = \boldsymbol{C}\boldsymbol{x}_{k+1}, \tag{62}$$

that can be described as follows:

$$\boldsymbol{z}_{k+1} = \boldsymbol{F}\boldsymbol{z}_k + \boldsymbol{TB}\boldsymbol{u}_k + \boldsymbol{K}_1\boldsymbol{y}_k, \tag{63}$$

$$\hat{\boldsymbol{x}}_{k+1} = \boldsymbol{z}_{k+1} + \boldsymbol{H}_1\boldsymbol{y}_{k+1}, \tag{64}$$

where

$$\boldsymbol{K}_1 = \boldsymbol{K} + \boldsymbol{K}_2, \tag{65}$$

$$\boldsymbol{E} = \boldsymbol{H}_1\boldsymbol{CE}, \tag{66}$$

$$\boldsymbol{T} = \boldsymbol{I} - \boldsymbol{H}_1\boldsymbol{C}, \tag{67}$$

$$\boldsymbol{F} = \boldsymbol{TA} - \boldsymbol{KC}. \tag{68}$$

Substituting (64) into (63) and then using (67) and (68) it can be shown that

$$\hat{\boldsymbol{x}}_{k+1} = \boldsymbol{A}\hat{\boldsymbol{x}}_k + \boldsymbol{B}\boldsymbol{u}_k - \boldsymbol{H}_1\boldsymbol{C}[\boldsymbol{A}\hat{\boldsymbol{x}}_k + \boldsymbol{B}\boldsymbol{u}_k] - \boldsymbol{K}\boldsymbol{C}\hat{\boldsymbol{x}}_k - \boldsymbol{F}\boldsymbol{H}_1\boldsymbol{y}_k + [\boldsymbol{K} + \boldsymbol{F}\boldsymbol{H}_1]\boldsymbol{y}_k + \boldsymbol{H}_1\boldsymbol{y}_{k+1}, \quad (69)$$

or equivalently:

$$\hat{\boldsymbol{x}}_{k+1} = \hat{\boldsymbol{x}}_{k+1/k} + \boldsymbol{H}_1(\boldsymbol{y}_{k+1} - \boldsymbol{C}\hat{\boldsymbol{x}}_{k+1/k}) + \boldsymbol{K}(\boldsymbol{y}_k - \boldsymbol{C}\hat{\boldsymbol{x}}_k), \quad (70)$$

where

$$\hat{\boldsymbol{x}}_{k+1/k} = \boldsymbol{A}\hat{\boldsymbol{x}}_k + \boldsymbol{B}\boldsymbol{u}_k. \quad (71)$$

Substituting the solution of (66), i.e. $\boldsymbol{H}_1 = \boldsymbol{E}\boldsymbol{H}$ into (70) yields:

$$\hat{\boldsymbol{x}}_{k+1} = [\boldsymbol{I} - \boldsymbol{E}\boldsymbol{H}\boldsymbol{C}]\hat{\boldsymbol{x}}_{k+1/k} + \boldsymbol{E}\boldsymbol{H}\boldsymbol{y}_{k+1} + \boldsymbol{K}(\boldsymbol{y}_k - \boldsymbol{C}\hat{\boldsymbol{x}}_k). \quad (72)$$

Thus, in order to use (72) for (56)-(57) it is necessary to change (71) into

$$\hat{\boldsymbol{x}}_{k+1/k} = \boldsymbol{A}\hat{\boldsymbol{x}}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{h}(\boldsymbol{y}_k, \boldsymbol{u}_k) + \boldsymbol{g}(\hat{\boldsymbol{x}}_k, \boldsymbol{u}_k). \quad (73)$$

Finally, by substituting (73) into (72) and then comparing it with (61) it can be seen that the observer structures being considered are identical. On the other hand, it should be clearly pointed out that they were designed in a significantly different way.

Since the observer structure is established then it is possible to describe its design procedure.

A simple comparison of (4) and (60) leads to the conclusion that the observer (61) can be designed with one of the techniques proposed in Section 4, taking into account that (cf. (3)):

$$\|\bar{\boldsymbol{g}}(\boldsymbol{x}_1, \boldsymbol{u}) - \bar{\boldsymbol{g}}(\boldsymbol{x}_2, \boldsymbol{u})\|_2 \le \bar{\gamma}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_2, \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}, \quad (74)$$

and assuming that the pair $(\bar{\boldsymbol{A}}, \boldsymbol{C})$ is observable.

6. Experimental results

6.1. Observer design and state estimation. The main objective of the present section is to compare the performance of the three different design procedures (proposed in Section 4).

First, the problem is to obtain the gain matrix $\boldsymbol{K}$ maximizing γ (for which the observer (6) is convergent) for the systems given by:

$$\boldsymbol{A} = \begin{bmatrix} 0.2 & 0.01 \\ 0.1 & 0.2 \end{bmatrix}, \quad \boldsymbol{C} = [1\,0], \quad (75)$$

and

$$\boldsymbol{A} = \begin{bmatrix} 0.137 & 0.199 & 0.284 \\ 0.0118 & 0.299 & 0.47 \\ 0.894 & 0.661 & 0.065 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \quad (76)$$

To tackle this problem the approaches presented in Section 4 were implemented with MATLAB®. One of the functions that implements the approach described in Section 4.2 is presented and carefully discussed in Appendix.

Table 1
Maximum γ for (75) and (76)

Design procedure	γ for (75)	γ for (76)
1	0.6765	0.5563
2	0.7998	0.6429
3	0.7916	0.5422

The obtained results are presented in Table 1. It can be observed that the maximum difference between the maximum Lipschitz constant obtained with the proposed design procedures is greater than 15%. Apart from the fact that the second design procedure gave the best results, it is probably impossible to prove that this is the best choice for all systems. The above results confirm Remark 1, i.e. it is very hard to chose a priori a criterion that gives the least conservative bound of γ. It is also worth to note that, contrary to the approaches presented in the literature (see e.g., [29–31,33]) the proposed procedures provide the gain matrix $\boldsymbol{K}$ that is a global solution of γ maximization problem.

Now let us consider a one-link manipulator with revolute joints actuated by a DC motor [30] described by (1)-(2) with the following parameters:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 10 \\ 1.95 & 0 & -1.95 & 0 \end{bmatrix},$$

$$\boldsymbol{B} = [0\,21.6\,0\,0]^T, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = [0\,0\,0\,\, -0.333\sin(x_3)], \;\; \boldsymbol{h}(\boldsymbol{y}(t), \boldsymbol{u}(t)) = \boldsymbol{0}, \quad (77)$$

where $x_1(t)$ stands for the angular rotation of the motor, $x_2(t)$ is the angular velocity of the motor, $x_3(t)$ is the angular position of the link, and $x_4(t)$ is the angular velocity of the link.

The discrete-time counterpart (4)-(5) of (77) was obtained by using the Euler discretization of a step size $\tau = 0.01$. The input signal was given by $u_k = \sin(2\pi\tau k)$ while the initial condition for the observer and the system were $\hat{\boldsymbol{x}}_0 = \boldsymbol{1}$ and $\boldsymbol{x}_0 = \boldsymbol{0}$, respectively.

The first objective was to compare the performance of the three different design procedures (proposed in Section 4). In particular, the problem was to obtain the gain matrix $\boldsymbol{K}$ maximizing γ.

As can be easily observed, the Lipschitz constant $\gamma = \tau 0.333$. The following maximum values of γ were obtained for the consecutive design procedures (i.e. 1, 2, 3): $\gamma = 0.0329$, $\gamma = 0.0802$, $\gamma = 0.0392$. This means that the acceptable γ (provided by the second design procedure) is more than 24 times larger than the actual $\gamma = 0.00333$. Similarly as in the preceding examples, the best results were achieved for the second design procedure. The resulting gain matrix $\boldsymbol{K}$ is:

$$\boldsymbol{K} = \begin{bmatrix} 1.0000 & 0.0100 \\ -0.4860 & 1.7926 \\ 0 & 1.9822 \\ 0.0195 & 3.2371 \end{bmatrix}. \tag{78}$$

For the purpose of comparison, a continuous-time observer:

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}(t) &= \boldsymbol{A}\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{g}\left(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)\right) \\ &\quad + \boldsymbol{K}(\boldsymbol{y}(t) - \boldsymbol{C}\hat{\boldsymbol{x}}(t)),\end{aligned} \tag{79}$$

designed by Rajamani and Cho [30] was employed. They obtained the following gain matrix $\boldsymbol{K}$ for (77):

$$\boldsymbol{K} = \begin{bmatrix} 0.8307 & 0.4514 \\ 0.4514 & 6.2310 \\ 0.8238 & 1.3072 \\ 0.0706 & 0.2574 \end{bmatrix}, \tag{80}$$

which is also utilized in this paper.

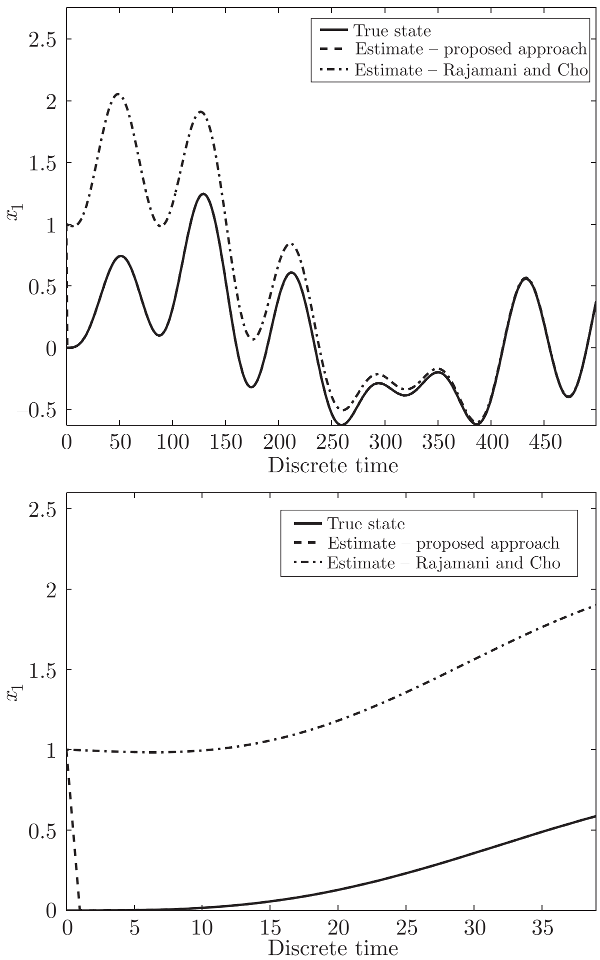

Fig. 1. Angular rotation of the motor x_1 and their estimates

Figures 1–4 show the results of state estimation. As can be observed, the state estimates obtained with the proposed observer converge rapidly to the corresponding true values (compare especially the estimates for $k = 0, \ldots, 40$ exposed by the plots on the right in Figs. 1–4). Indeed, it can be easily seen that the proposed observer

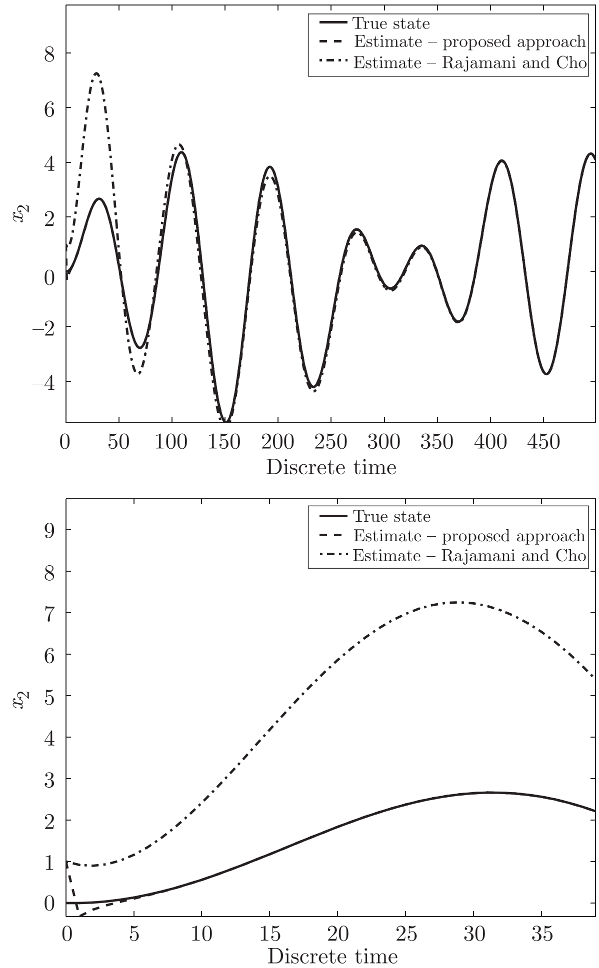

Fig. 2. Angular velocity of the motor x_2 and their estimates

is superior to the one designed with the design procedure proposed in [30]. This superiority can be clearly seen in Fig. 5, which exposes the evolution of the norm of the state estimation error.

6.2. Design of UIO and fault detection. Let us consider the following nonlinear system:

$$\begin{aligned}\boldsymbol{x}_{k+1} &= \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\left(\boldsymbol{u}_k + \boldsymbol{f}_k\right) + \boldsymbol{g}\left(\boldsymbol{x}_k\right) + \boldsymbol{E}\boldsymbol{d}_k, \\ \boldsymbol{y}_{k+1} &= \boldsymbol{C}\boldsymbol{x}_{k+1},\end{aligned}$$

where:

$$\boldsymbol{A} = \begin{bmatrix} 0.137 & 0.199 & 0.284 \\ 0.0118 & 0.299 & 0.47 \\ 0.894 & 0.661 & 0.065 \end{bmatrix}, \boldsymbol{B} = [0.25\, 0.6\, 0.1]^T,$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \boldsymbol{E} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\boldsymbol{g}\left(\boldsymbol{x}_k\right) = \left[0.6\frac{\cos(12x_{2,k})}{x_{2,k}^2 + 10} \quad 0 \ -0.333\sin(x_{3,k})\right]^T,$$

and $\boldsymbol{f}_k$ stands for the actuator fault, which is given as follows:

$$\boldsymbol{f}_k = \begin{cases} -0.1\boldsymbol{u}_k, & 50 \le k \le 150 \\ 0, & \text{otherwise} \end{cases}.$$

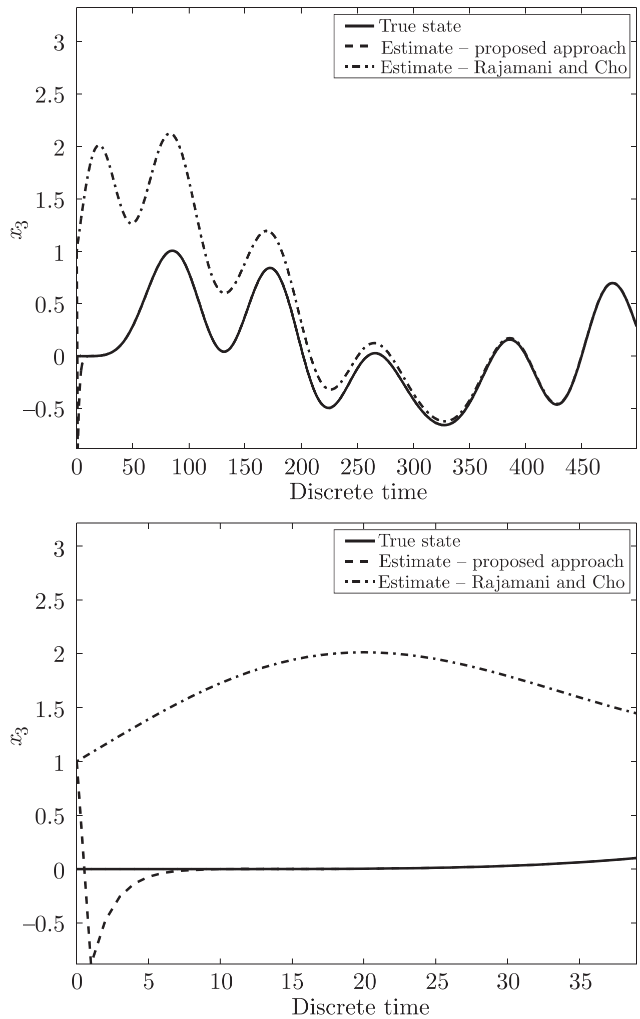

Fig. 3. Angular position of the link x_3 and their estimates

The initial condition for the system and the observer are $\boldsymbol{x}_0 = [3\,2\,1]^T$ and $\hat{\boldsymbol{x}}_0 = \boldsymbol{0}$, respectively. Moreover, the input and unknown input are given by $\boldsymbol{u}_k = \sin(0.02\pi k)$ and $\boldsymbol{d}_k = 0.3\sin(0.1k)\cos(0.2k)$, respectively.

Applying the approach presented in Section 5, it can be observed that $\bar{\gamma} = \gamma$. Following the general approach for estimating the Lipschitz constant [28], one can show that $\bar{\gamma} = \gamma = 0.719$. Knowing the Lipschitz constant it is possible to use the procedures presented in Section 4 to design the UIO. The maximum allowable γ for the consecutive (1, 2 and 3) design procedures were 0.65, 0.772 and 0.722, respectively. This means that the observer (the gain matrix $\boldsymbol{K}$) obtained with the first procedure cannot be employed because the maximum $\bar{\gamma}$ for which the observer is convergent is $\bar{\gamma} = 0.65$. Indeed, it is smaller than the actual value $\bar{\gamma} = 0.719$.

Thus, it can be seen that the second design procedure is less restrictive for the system being considered. The similar property has been observed for a large number of numerical examples. For the purpose of comparison, a conventional observer was designed with the use of the second procedure, i.e. the effect of an unknown input was neglected during the design. Figures 6–7, present the residual for the UIO and the conventional observer. As can be observed, this is impossible to detect the actuator fault with the conventional observer and the fixed threshold (presented in the figure). Contrary, it is straightforward to assign a fixed threshold to the residual generated with the UIO, and then to detect the actuator fault with $z_{1,k} = y_{1,k} - \hat{y}_{1,k}$ (Fig. 6).

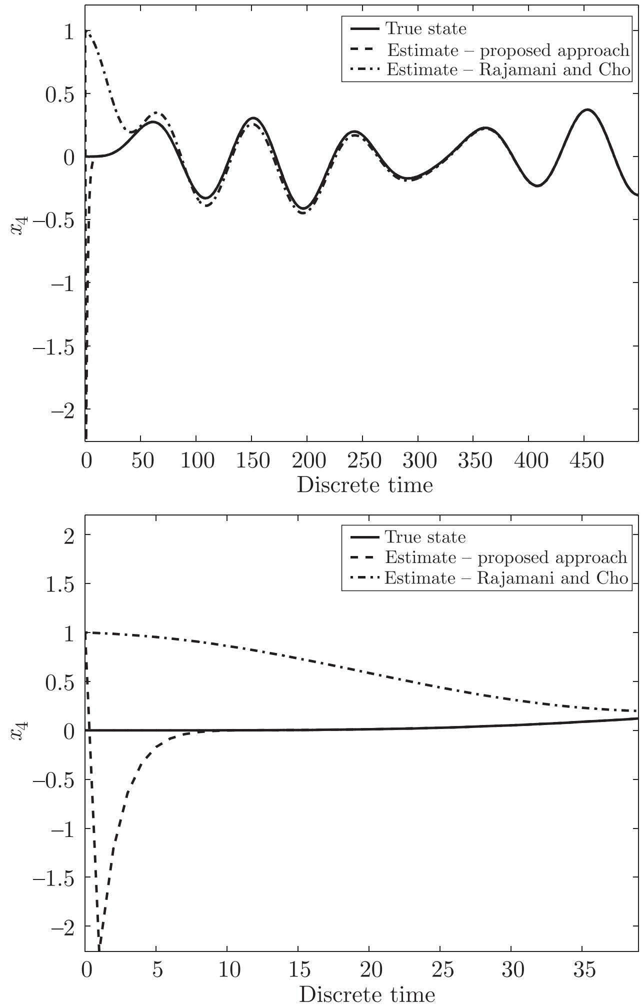

Fig. 4. Angular velocity of the link x_4 and their estimates

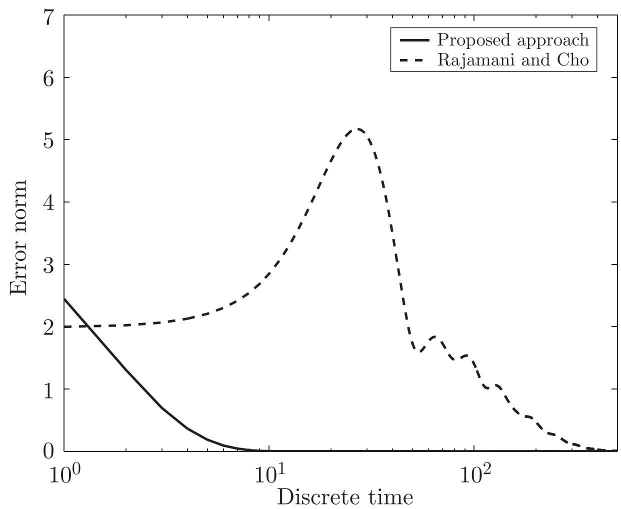

Fig. 5. Norm of the state estimation error $\|\boldsymbol{e}_k\|_2$

7. Conclusions

The main objective of this paper was to develop efficient approaches to designing observers for discrete-time Lipschitz non-linear systems. In particular, with the use of the Lyapunov method, three different convergence criteria were developed. The difference between these criteria lies in the way the Lyapunov function is calculated. All these techniques introduce a level of conservatism related to the relative inaccuracy of a given technique. Based on the achieved results, three different design procedures were proposed. These procedures were developed in such a way as the design problem boils down to solving a set of linear matrix inequalities or solving the generalized eigenvalue minimization problem under LMI constraints, respectively. Experimental results confirm the effectiveness of the proposed design procedures. In particular, it was shown that the proposed approach can be effectively applied to design an observer for a flexible link robot. Moreover, the convergence rate provided by the proposed observer is significantly higher that the one obtained with the techniques presented in the literature.

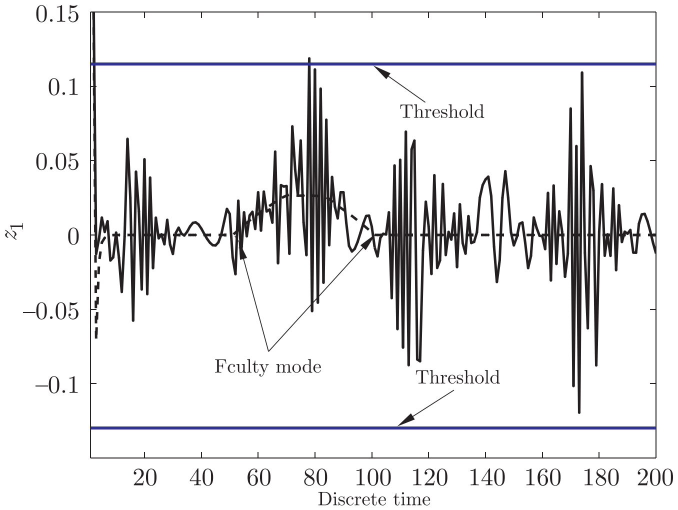

Fig. 6. Residual $z_{1,k} = y_{1,k} - \hat{y}_{1,k}$ obtained with the UIO (dashed line) and the conventional observer

Another objective of the paper was to show how to apply the proposed techniques for the systems with unknown inputs. This was realized with the use of a suitable system transformation that converts the system description with an unknown input into the system description without it. Experimental results confirm the effectiveness of the proposed design procedures and show the potential profits that can be achieved while applying the proposed approach in the FDI scheme.

The author hopes that the presented results as well as the developed MATLAB® procedure will encourage engineers to apply the proposed techniques in practice.

Appendix

The main objective of this appendix is to show the implementation details regarding design procedures described in Section 4. Since the design procedures are very similar, the attention is restricted to the implementation of one of them, namely the approach described in Section 4.2. The procedures presented in Sections 4.1 and 4.3 can easily be implemented by a minor modification of the the source code presented in this appendix.

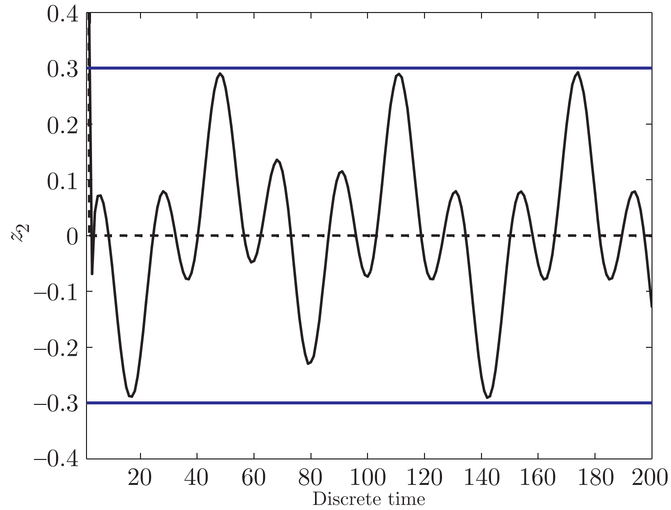

Fig. 7. Residual $z_{2,k} = y_{2,k} - \hat{y}_{2,k}$ obtained with the UIO (dashed line) and the conventional observer

In particular, the selection of $\boldsymbol{K}$ maximizing γ for which the observer (6) is convergent is considered. Figure 8 presents the complete MATLAB® source code that can be used to solve such a problem. As has already been mentioned in Section 4.1, the positivity of the right hand side of (45) is required for the well-posedness of the task and the applicability of the polynomial-time interior point methods [36]. To tackle this problem, a simple remedy described in [36] is employed. As a result, instead of using (45), the following two LMIs are utilized:

$$\begin{bmatrix} \boldsymbol{X}-\boldsymbol{P} & \boldsymbol{C}^T\boldsymbol{L}^T-\boldsymbol{A}^T\boldsymbol{P} \\ \boldsymbol{L}\boldsymbol{C}-\boldsymbol{P}\boldsymbol{A} & -\boldsymbol{P} \end{bmatrix} \prec \begin{bmatrix} \boldsymbol{Y} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \tag{81}$$

and

$$\boldsymbol{Y} \prec \lambda(\beta+1)\boldsymbol{I}, \tag{82}$$

where $\boldsymbol{Y}$ is a symmetric matrix.

Now, let us describe the function presented in Fig. 6. The input and output parameters are $\boldsymbol{A}$, $\boldsymbol{C}$ and γ, $\boldsymbol{K}$, respectively. In lines 2-4, all variables required in LMIs (33), (43), (81) and (82) are set. LMIs (33) are defined in lines 5-7, while lines 8-9 implement (43). Finally, LMIs (81) and (82) are defined in lines 10-14. Since all LMIs are defined, it is possible to find $\boldsymbol{K}$ maximizing γ for which the observer (6) is convergent. As has been described in Section 4.2, such an optimization task is formulated as a generalized eigenvalue minimization problem [35,36]. The code of lines 15-16 is employed to solve this problem. In line 17, the correctness of the achieved solution is checked. Finally, in lines 18-19 the maximum γ for which the observer (6) is convergent and the corresponding $\boldsymbol{K}$ are calculated.

```
function [gamma,K]=GetGamma2(A,C)
n=size(A,1); m=size(C,1); setlmis([]);
P=lmivar(1,[n 1]); L=lmivar(2,[n m]); X=lmivar(1,[n 1]);
Y=lmivar(1,[n 1]); Beta=lmivar(1,[1 1]);
lmiterm([-1 1 1 P],1,1); lmiterm([-2 1 1 Beta],1,1);
lmiterm([-3 1 1 Beta],1,1);lmiterm([-3 2 2 Beta],1,1);
lmiterm([-3 2 1 P],1,1);
lmiterm([-4 1 1 X],1,1); lmiterm([-4 2 2 0],1);
lmiterm([-4 2 1 P],1,A); lmiterm([-4 2 1 L],-1,C);
lmiterm([5 1 1 P],-1,1); lmiterm([5 2 2 P],-1,1);
lmiterm([5 2 1 P],-1,A); lmiterm([5 2 1 L],1,C);
lmiterm([5 1 1 Y],-1,1); lmiterm([5 1 1 X],1,1);
lmiterm([6 1 1 Y],1,1); lmiterm([-6 1 1 Beta],1,1);
lmiterm([-6 1 1 0],1);
LMIss=getlmis;
[lambda, popt]=gevp(LMIss,1);
if lambda>0 error('The problem cannot be solved'); end;
gamma=sqrt(-lambda); P=dec2mat(LMIss,popt,P);
L=dec2mat(LMIss,popt,L); K=inv(P)*L;
```

Fig. 8. MATLAB® implementation of the design procedure of Section 4.2

Acknowledgements. The work was supported by the Ministry of Science and Higher Education in Poland.

References

[1] J. Korbicz, J.M. Kościelny, Z. Kowalczuk, and W. Cholewa, *Fault Diagnosis. Models, Artificial Intelligence, Applications*, Springer-Verlag, Berlin 2004.

[2] J. Chen and R. J. Patton, *Robust Model-based Fault Diagnosis for Dynamic Systems*, Kluwer Academic Publishers, London, 1999.

[3] F. Delebecque, R. Nikoukah, and H. Rubio Scola, "Test signal design for failure detection: a linear programming approach", *Int. J. Appl. Math. Comput. Sci.* 13 (4), 515–526 (2003).

[4] Z. Kowalczuk and T. Bialaszewski, "Niching mechanisms in evolutionary computation", *Int. J. Appl. Math. Comput. Sci.* 16 (1), 59–84 (2006).

[5] E. Alcorta Garcia and P.M. Frank, "Deterministic nonlinear observer-based approaches to fault diagnosis", *Control. Eng. Practice* 5 (5), 663–670 (1997).

[6] C. Califano, S. Monaco, and D. Normand-Cyrot, "On the observer design in discrete-time", *System and Control Letters* 49, 255–265 (1997).

[7] M. Witczak, A. Obuchowicz, and J. Korbicz, "Genetic programming based approaches to identfication and fault diagnosis of non-linear dynamic systems", *Int. J. Contr.* 75 (13), 1012–1031 (2002).

[8] M. Witczak, *Identification and Fault Detection of Nonlinear Dynamic Systems*, University of Zielona Góra Press, Zielona Góra, 2003.

[9] R. Seliger and P. Frank, "Robust observer-based fault diagnosis in non-linear uncertain Systems", in *Issues of Fault Diagnosis for Dynamic Systems*, edited by R. J. Patton, P. Frank, and R. N. Clark, Springer-Verlag, Berlin, 2000.

[10] S.H. Wang, E.J. Davison, and P. Dorato, "Observing the states of systems with unmeasurable disturbances", *IEEE Trans. Automatic Control* 20 (5), 716–717 (1975).

[11] S.P. Bhattacharyya, "Observer design for linear systems with unknown inputs", *IEEE Trans. Automat. Contr.* 23 (3), 483–484 (1978).

[12] N. Kobayashi and R. Nakamizo, "An observer design for linear systems with unknown inputs", *Int. J. Contr.* 17 (3), 471–479 (1982).

[13] M. Hou and P.C. Müller, "Design of observers for linear systems with unknown inputs", *IEEE Trans. Automatic Control* 37 (6), 871–875 (1992).

[14] J. Chen, R. J. Patton, and H. Zhang, "Design of unknown input observers and fault detection filters" — *Int. J. Contr.* 63 (1), 85–105 (1996).

[15] S. Hui, S.H. Zak, "Observer design for systems with unknown input", *Int. J. Appl. Math. Comput. Sci.* 15 (4), 431–446 (2005).

[16] M. Darouach and M. Zasadzinski, "Unbiased minimum variance estimation for systems with unknown exogenous inputs", *Automatica* 33 (4), 717–719 (1997).

[17] M. Hou and R.J. Patton, "Optimal filtering for systems with unknown inputs", *IEEE Trans. Automatic Control* 43 (3), 445–449 (1998).

[18] J. Y. Keller and M. Darouach, "Two-stage Kalman estimator with unknown exogenous inputs", *Automatica* 35, 339–342 (1999).

[19] D. Maksarov and J. Norton, "State bounding with ellipsoidal set description of the uncertainty" *Int. J. of Control* 65 (5), 847–866 (1996).

[20] S.A. Ashton, D.N. Shields, and S. Daley, "Design of a robust fault detection observer for polynomial nonlinearities", *Proc. 14th IFAC World Congress*, Beijing, China, CD-ROM (1999).

[21] A. Hac, "Design of disturbance decoupled observers for bilinear systems", *ASME J. Dynamic Sys. Measure. Contr.* 114, 556–562 (1992).

[22] D. Koenig and S. Mammar, "Design of a class of reduced unknown inputs non-linear observer for fault dagnosis", *Proc. American Control Conference*, Arlington, USA, CD-ROM (2001).

[23] A.M. Pertew, H.J. Marquez, and Q. Zhao, "Design of unknown input observers for Lipschitz non-linear systems", *Proc. American Control Conference*, Portland, USA, CD-ROM (2005).

[24] A. Zolghardi, D. Henry, and M. Monision, "Design of nonlinear observers for fault diagnosis. A case study", *Control. Eng. Practice* 4 (11), 1535–1544 (1996).

[25] M. Boutayeb and D. Aubry, "A strong tracking extended Kalman observer for non-linear discrete-time systems", *IEEE Trans. Automat. Contr.* 44 (8), 1550–1556 (1999).

[26] Y. Song and J.W. Gizzle, "The extended Kalman filter as a local asymptotic observer for non-linear discrete-time systems", *J. Math., Estimation, Contr.* 5, 59–78 (1995).

[27] F.E. Thau, "Observing the state of non-linear dynamic systems", *Int. J. Contr* 17 (3), 471–479 (1973).

[28] G. Schreier, J. Ragot, R.J. Patton, and P.M. Frank, "Observer design for a class of non-linear systems", *IFAC Symp.: Fault Detection, Supervision and Safety of Technical Processes: SAFEPROCESS'97* 1, 483–488 (1997).

[29] C. Aboky, G. Sallet, and J.C. Vivalda, "Observers for Lipschitz non-linear systems", *Int. J. Contr.* 75, (3), 204–212, (2002).

[30] R. Rajamani and Y.M. Cho, "Existence and design of observers for non-linear systems: relation to distance to unobservability", *Int. J. Contr.* 69 (5), 717–731 (1998).

[31] R. Rajamani, "Observers for Lipschitz non-linear systems", *IEEE Trans. Aut. Control* 43 (3), 397–401 (1998).

[32] K. Busawon, M. Saif, and M. Farza, "A discrete-time observer for a class of non-linear systems", *36th IEEE Conference on Decision and Control*, 4796–4801 (1997).

[33] Z. Wang and H. Unbehauen "A class of non-linear observers for discrete-time systems with parametric uncertainty", *Int. J. Sys. Sci.* 31 (1), 19–26 (2000).

[34] H.G. Chen and K.W. Han, "Improved quantitative measures of robustness for multivariable systems", *IEEE Trans. Automatic Control* 39 (4), 807–810 (1994).

[35] S. Boyd, E. Feron, L.E. Ghaoui, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Siam, Philadelphia, 1994.

[36] P. Gahinet, A. Nemirovski, A.J. Laub, and M. Chilali, *LMI Control Toolbox. For Use with Matlab*, MathWorks Inc., Natick, 1995.