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Robust stability of positive discrete-time linear systems with multiple delays with linear unity rank uncertainty structure or non-negative perturbation matrices

M. BUSŁOWICZ*

Faculty of Electrical Engineering, Technical University of Białystok, 45D Wiejska St., 15-351 Białystok, Poland

Abstract. Simple necessary and sufficient conditions for robust stability of the positive linear discrete-time systems with delays with linear uncertainty structure in two cases: 1) unity rank uncertainty structure, 2) non-negative perturbation matrices, are established. The proposed conditions are compared with the suitable conditions for the standard systems. The considerations are illustrated by numerical examples.

Key words: robust stability, linear system, positive, discrete-time, delays, linear uncertainty, unity rank uncertainty.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values for non-negative initial states and non-negative controls. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc. [1,2]. Recently, conditions for stability and robust stability of positive discrete-time systems with delays were given in [3–11].

The main purpose of the paper is to give the simple necessary and sufficient conditions for robust stability of linear positive discrete-time systems with delays with linear uncertainty structure in two cases: 1) unity rank uncertainty structure, 2) non-negative perturbation matrices. The proposed conditions will be compared with the suitable conditions for the standard systems.

2. Problem formulation

Let $\Re_+^{n\times m}$ be the set of $n \times m$ matrices with real non-negative entries and $\Re_+^n = \Re_+^{n\times 1}$. The set of non-negative integers will be denoted by Z_+.

Consider an uncertain positive discrete-time linear system with delays described by the homogeneous equation

$$x_{i+1} = \sum_{k=0}^{h} A_k(q_k)x_{i-k}, \quad q_k \in Q_k, \quad i \in Z_+, \tag{1}$$

where h is a positive integer, $x_i \in \Re^n$ is the state vector,

$$A_k(q_k) = A_{k0} + \sum_{r=1}^{m_k} q_{kr}E_{kr}, \quad k = 0, 1, ..., h, \tag{2}$$

$A_{k0} \in \Re_+^{n\times n}$ and $E_{kr} \in \Re^{n\times n}$ $(k = 0, 1, ..., h,\ r = 1, 2, ..., m_k)$ are the nominal and the perturbation matrices, respectively, $q_k = [q_{k1}, q_{k2}, ..., q_{km_k}] \in Q_k$ is the k-th $(k = 0, 1, ..., h)$ sub-vector of uncertain parameters $q_{k1},\ q_{k2}, ..., q_{km_k}$ and

$$Q_k = \{q_k : q_{kr} \in [q_{kr}^-, q_{kr}^+],\ r = 1, 2, \dots, m_k\} \tag{3}$$

with $q_{kr}^- \leqslant 0,\ q_{kr}^+ \geqslant 0$ $(k = 0, 1, ..., h,\ r = 1, 2, ..., m_k)$ is the value set of these parameters.

Then $q = [q_0, q_1, ..., q_h]$ is the vector of uncertain parameters of the system (1) and

$$Q = Q_0 \times Q_1 \times \cdots \times Q_h \tag{4}$$

is the value set of uncertain parameters.

The perturbed system (1) is positive if and only if

$$A_k(q_k) \in \Re_+^{n\times n},\ \forall q_k \in Q_k\ (k = 0, 1, ..., h). \tag{5}$$

The condition (5) can be written in the form

$$a_{ij}^k + \sum_{r=1}^{m_k} \alpha_{ij}^{kr} \geqslant 0,\ i, j = 1, 2, ..., n, \tag{6a}$$

where

$$\alpha_{ij}^{kr} = \begin{cases} q_{kr}^- e_{ij}^{kr} \text{ if } e_{ij}^{kr} \geqslant 0 \\ q_{kr}^+ e_{ij}^{kr} \text{ if } e_{ij}^{kr} < 0 \end{cases} \tag{6b}$$

with $A_{k0} = [a_{ij}^k],\ E_{kr} = [e_{ij}^{kr}],\ i, j = 1, 2, ..., n,\ k = 0, 1, ..., h.$

Let us introduce the following assumptions.

Assumption 1. The system (1) has unity rank uncertainty structure, that is the following conditions hold

$$\text{rank}E_{kr} = 1 \quad \text{for} \quad k = 0, 1, ..., h,\ r = 1, 2, ..., m_k. \tag{7}$$

Assumption 2. The positive nominal system

$$x_{i+1} = \sum_{k=0}^{h} A_{k0}x_{i-k},\ i \in Z_+, \tag{8}$$

*e-mail: busmiko@pb.edu.pl

corresponding to the nominal values of uncertain parameters $q_{kr} = 0,\ k = 0, 1, ..., h,\ r = 1, 2, ..., m_k$ is asymptotically stable, that is all roots $z_1, z_2, ..., z_{\tilde{n}}$ with $\tilde{n} = (h+1)n$ of the characteristic equation $w(z) = 0$ have absolute values less than 1, where [11]

$$w(z) = \det(z^{h+1} I_n - \sum_{k=0}^{h} A_{k0} z^{h-k}). \tag{9}$$

The perturbed positive system (1) is robustly stable if and only if all roots $z_1(q),\ z_2(q), \dots z_{\tilde{n}}(q),\ \tilde{n} = (h+1)n$, of the characteristic equation

$$w(z,q) = \det(z^{h+1} I_n - \sum_{k=0}^{h} A_k(q_k) z^{h-k}) = 0 \tag{10}$$

satisfy the conditions $|z_i(q)| < 1,\ i = 1, 2, \dots, \tilde{n} = (h+1)n$, for all $q \in Q$.

In the paper we give simple necessary and sufficient conditions for robust stability of the positive discrete-time system (1) with delays with linear uncertainty structure in two cases:

1) unity rank uncertainty structure (the conditions (7) hold),
2) non-negative perturbation matrices, i.e. $E_{kr} \in \Re_+^{n \times n}$ for $k = 0, 1, ..., h,\ r = 1, 2, ..., m_k$ (satisfaction of (7) is not necessary).

3. The main results

The positive system without delays equivalent to (1) has the form

$$\tilde{x}_{i+1} = A(q)\tilde{x}_i,\ q \in Q, \tag{11}$$

where the state vector $\tilde{x}_i \in \Re_+^{\tilde{n}}$ with $\tilde{n} = (h+1)n$ and

$$A(q) = \begin{bmatrix} A_0(q_0) & A_1(q_1) & \cdots & A_h(q_h) \\ I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I_n & 0 \end{bmatrix} \in \Re_+^{\tilde{n} \times \tilde{n}}. \tag{12}$$

The positive system (11) is robustly stable if and only if

$$w_A(z,q) = \det(zI_{\tilde{n}} - A(q)) \neq 0 \quad \text{for} \quad |z| \geqslant 1,\ \forall q \in Q. \tag{13}$$

It is easy to see that $w(z,q) = w_A(z,q)$ (see for example [11] for the system without uncertain parameters). Hence, robust stability of the positive system (1) (with delays) is equivalent to robust stability of the positive system (11) (without delays).

Substituting $q = 0$ in (11), (12) and (2) we obtain that the positive system without delays equivalent to the nominal system (8) with delays has the form

$$\tilde{x}_{i+1} = A_0 \tilde{x}_i,\ i \in Z_+, \tag{14}$$

where $\tilde{x}_i \in \Re_+^{\tilde{n}},\ \tilde{n} = (h+1)n$ and

$$A_0 = \begin{bmatrix} A_{00} & A_{10} & \cdots & A_{h0} \\ I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I_n & 0 \end{bmatrix} \in \Re_+^{\tilde{n} \times \tilde{n}}. \tag{15}$$

From [11] (see also [6]) we have the following theorems and lemma.

THEOREM 1. The positive system (8) with delays is asymptotically stable if and only if the following equivalent conditions hold:
1) all coefficients of the characteristic polynomial of the matrix $S_0 = A_0 - I_{\tilde{n}}$, of the form

$$\begin{aligned} &\det[(z+1)I_{\tilde{n}-A_0}] \\ &= \det\left[(z+1)^{h+1} I_n - \sum_{k=0}^{h} A_{k0}(z+1)^{h-k}\right] \\ &= z^{\tilde{n}} + a_{\tilde{n}-1} z^{\tilde{n}-1} + \ldots + a_1 z + a_0 \end{aligned} \tag{16}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, ..., \tilde{n}-1$,
2) all leading principal minors of the matrix $\bar{A}_0 = I_{\tilde{n}} - A_0$ are positive.

THEOREM 2. The positive system with delays (8) is unstable if the positive system without delays

$$x_{i+1} = A_{00} x_i,\ i \in Z_+, \tag{17}$$

is unstable.

LEMMA 1. The positive system (17) is unstable if at least one diagonal entry of the matrix $A_{00} = \left[a_{ij}^0\right]$ is greater than 1, i.e. $a_{ii}^0 > 1$ for some $i \in (1, 2, ..., n)$.

By generalisation of Theorem 1 to the system (1) with uncertain parameters we obtain the following theorem.

THEOREM 3. The positive system with delays (1) is robustly stable if and only if the following equivalent conditions hold:
1) all coefficients of the characteristic polynomial of the matrix $S(q) = A(q) - I_{\tilde{n}}$ of the form

$$S(q) = \begin{bmatrix} A_0(q_0) - I_n & A_1(q_1) & \cdots & A_h(q_h) \\ I_n & -I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I_n & -I_n \end{bmatrix}, \tag{18}$$

are positive for all $q \in Q$,
2) all leading principal minors $\Delta_i(q)\ (i = 1, 2, ..., \tilde{n})$ of the matrix $\bar{A}(q) = I_{\tilde{n}} - A(q) = -S(q)$ of the form

$$\bar{A}(q) = \begin{bmatrix} I_n - A_0(q_0) & -A_1(q_1) & \cdots & -A_h(q_h) \\ -I_n & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -I_n & I_n \end{bmatrix}, \tag{19}$$

are positive for all $q \in Q$.

From (19) it follows that positivity of all leading principal minors of the matrix $I_n - A_0(q_0),\ q_0 \in Q_0$, is necessary for positivity of all leading principal minors of the matrix $\bar{A}(q)$ for all $q \in Q$. This means that robust stability of the positive system without delays

$$x_{i+1} = A_0(q_0)x_i,\ q_0 \in Q_0, \tag{20}$$

is necessary for robust stability of the positive system (1).

From the above we have the following generalisation of Theorem 2 to the positive system (1) with uncertain parameters.

THEOREM 4. Robust stability of the positive system (20) (without delays) is necessary for robust stability of the positive system (1) (with delays).

From generalisation of Lemma 1 it follows that the positive system (20) is not robustly stable if there exists $q_0 \in Q_0$ such that at least one diagonal entry of the matrix $A_0(q_0)$ is greater than 1.

Let us denote by $\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_L$ ($L = 2^m,\ m = m_0 + m_1 + \ldots + m_h$), where $\bar{q}_l = \left[q_0^l, q_1^l, \ldots, q_h^l\right]$ and $q_k^l = [q_{k1}^l, q_{k2}^l, ..., q_{km_k}^l]$ with $q_{kr}^l = q_{kr}^-$ or $q_{kr}^l = q_{kr}^+$, $k = 0, 1, ..., h,\ r = 1, 2, ..., m_k$ the vertices of hiperrectangle (4).

Moreover, by $A_l^v = A(\bar{q}_l),\ l = 1, 2, ..., L$, denote the vertex matrices of the family of non-negative matrices $\{A(q) : q \in Q\}$ where $A(q)$ has the form (12). These matrices correspond to the vertices of the set (4).

THEOREM 5. The positive system (1) with linear unity rank uncertainty structure is robustly stable if and only if the finite family of positive systems

$$\tilde{x}_{i+1} = A_l^v \tilde{x}_i,\ l = 1, 2, ..., L = 2^m, \tag{21}$$

is asymptotically stable, i.e. the conditions of Theorem 3 are satisfied for all $q = \bar{q}_l,\ l = 1, 2, ..., L$.

Proof. *Necessity.* Necessity is obvious because the systems (21) belong to the family (11) of the positive systems.

Sufficiency. Characteristic polynomial of the matrix (18) can be written in the form

$$\det(zI_{\tilde{n}} - (A(q) - I_{\tilde{n}})) = z^{\tilde{n}} + \sum_{i=0}^{\tilde{n}-1} \tilde{a}_i(q) z^i. \tag{22}$$

It is easy to see that if Assumption 1 holds then the matrix $A(q)$ of the form (12) has linear unity rank uncertainty structure. Hence, the coefficients $\tilde{a}_i(q),\ i = 0, 1, ..., \tilde{n} - 1$, of (22) are real multilinear functions of uncertain parameters $q_{kr},\ k = 0, 1, ..., h,\ r = 1, 2, ..., m_k$ [12] and

$$\min_{q \in Q} \tilde{a}_i(q) = \min_{l} \tilde{a}_i(\bar{q}_l),\ i = 0, 1, ..., \tilde{n} - 1. \tag{23}$$

From the condition 1) of Theorem 1 it follows that if the family (21) of the positive systems is asymptotically stable, then all coefficients of the characteristic polynomials of the matrices $S(\bar{q}_l) = A_l^v - I_{\tilde{n}},\ l = 1, 2, ..., L$, are positive, i.e.

$$\tilde{a}_i(\bar{q}_l) > 0,\ i = 0, 1, ..., \tilde{n} - 1,\ l = 1, 2, ..., L. \tag{24}$$

Hence, $\min_{l} \tilde{a}_i(\bar{q}_l) > 0$ for $i = 0, 1, ..., \tilde{n} - 1$, and by (23),

$$\min_{q \in Q} \tilde{a}_i(q) > 0,\ i = 0, 1, ..., \tilde{n} - 1. \tag{25}$$

This means that all coefficients of the polynomial (22) are positive for all $q \in Q$, and by condition 1) of THEOREM 3, the positive system (1) is robustly stable.

To asymptotic stability analysis of the positive systems (21) we can apply Theorem 1 putting $A_l^v = A(\bar{q}_l)$ for $l = 1, 2, ..., L$, instead of the matrix A_0.

LEMMA 2. Asymptotic stability of the positive systems

$$x_{i+1} = A_0^- x_i \quad \text{and} \quad x_{i+1} = A_0^+ x_i, \tag{26}$$

where $A_0^- = A_{00} + \sum_{r=1}^{m_0} q_{0r}^- E_{0r},\ A_0^+ = A_{00} + \sum_{r=1}^{m_0} q_{0r}^+ E_{0r}$ is necessary for robust stability of the positive system (1) with uncertain parameters.

Proof. In the same way as in the proof of Theorem 5 we can show that the positive system (20) with linear unity rank uncertainty structure is robustly stable if and only if the positive systems (26) are asymptotically stable. Hence, the proof follows directly from Theorem 4.

LEMMA 3. The perturbed positive system (1) is not robustly stable if at least one diagonal entry of the matrix $A_0(q_0)$ is greater than 1 for some $q_0 \in Q_0$.

Proof. It follows directly form Theorem 4 and generalisation of Lemma 1 to the positive system (20).

Now consider the positive system (1) with non-negative perturbation matrices, i.e. with $E_{kr} \in \Re_+^{n \times n}$, $k = 0, 1, ..., h,\ r = 1, 2, ..., m_k$. In such a case $q_{kr}E_{kr} \in \left[q_{kr}^- E_{kr}, q_{kr}^+ E_{kr}\right]$ for any fixed $q_{kr} \in \left[q_{kr}^-, q_{kr}^+\right]$. Therefore, robust stability of the positive system (1) with non-negative perturbation matrices is equivalent to robust stability of the positive interval system

$$x_{i+1} = \sum_{k=0}^{h} A_k x_{i-k},\ A_k \in \left[A_k^-, A_k^+\right] \subset \Re_+^{n \times n} \tag{27}$$

where

$$A_k^- = A_{k0} + \sum_{r=1}^{m_k} q_{kr}^- E_{kr},\ A_k^+ = A_{k0} + \sum_{r=1}^{m_k} q_{kr}^+ E_{kr} \tag{28}$$

for $k = 0, 1, ..., h$.

Moreover, for non-negative perturbation matrices $E_{kr} (k = 0, 1, ..., h,\ r = 1, 2, ..., m_k)$, from (12) and (2) we have that $A(q) \in [A^-, A^+]$ for all $q \in Q$, where $A^- = A(q^-),\ A^+ = A(q^+)$ with $q^- = \left[q_0^-, q_1^-, ..., q_h^-\right]$ and $q^+ = \left[q_0^+, q_1^+, ..., q_h^+\right]$, i.e.

$$A^- = \begin{bmatrix} A_0^- & A_1^- & \cdots & A_h^- \\ I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I_n & 0 \end{bmatrix},\ A^+ = \begin{bmatrix} A_0^+ & A_1^+ & \cdots & A_h^+ \\ I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & I_n & 0 \end{bmatrix}, \tag{29}$$

where A_k^- and A_k^+ are computed from (28).

It is easy to see that interval matrix $[A^-, A^+]$ is non-negative if and only if $A^- \in \Re_+^{\tilde{n}\times\tilde{n}}$.

From the above it follows that robust stability of the positive system (1) with linear uncertainty structure and non-negative perturbation matrices is equivalent to robust stability of the interval positive discrete-time system without delays

$$\tilde{x}_{i+1} = A\tilde{x}_i,\ A \in \left[A^-, A^+\right] \subset \Re_+^{\tilde{n}\times\tilde{n}} \tag{30}$$

The robust stability problem of the positive discrete-time interval systems with delays was considered in [6]. From the above and [6] we have the following Theorem and Lemma.

Theorem 6. The positive system (1) with non-negative perturbation matrices is robustly stable if and only if the positive system without delays

$$\tilde{x}_{i+1} = A^+\tilde{x}_i,\ i \in Z_+, \tag{31}$$

where A^+ has the form given in (29), is asymptotically stable, i.e. the conditions of Theorem 3 hold for $q = q^+ = [q_0^+, q_1^+, ..., q_h^+]$.

Lemma 4. The positive system (1) with non-negative perturbation matrices is not robustly stable if at least one diagonal entry of the matrix A_0^+ is greater than 1.

4. Comparison of robust stability conditions for positive and standard systems

The proposed conditions for robust stability of the positive system (1) with delays are expressed in terms of the equivalent positive discrete-time system without delays (11).

In this section we compare the proposed conditions of robust stability of the positive system (11) with the suitable conditions for the standard system (11), i.e. without assumption that $A(q) \in \Re_+^{\tilde{n}\times\tilde{n}}$ for all $q \in Q$.

The problem of robust stability analysis of standard linear systems without delays, with structured uncertainty in the state-space description or in the characteristic polynomials was considered in many papers and books, see monographs [12–16], for example. These papers are mainly directed to the robust stability problems of characteristic polynomials with linear and multilinear uncertainty structure.

In the case of the standard (or positive) systems with linear unity rank uncertainty structure the coefficients of the characteristic polynomial $w_A(z,q) = \det(zI_{\tilde{n}} - A(q))$ are multilinear functions of uncertain parameters [14]. Therefore, applying the Mapping Theorem of Zadeh and Desoer and the Edge Theorem (see [12–14,16], for example) we obtain the following sufficient condition for robust stability of the standard system (11).

Theorem 7. The standard system (11) with linear unity rank uncertainty structure is robustly stable if the vertex standard systems (21) are asymptotically stable and the following set of one parameter standard systems

$$\tilde{x}_{i+1} = A_{jk}(\lambda)\tilde{x}_i,\ j,k = 1,2,...,L = 2^m,\ j > k, \tag{32}$$

is robustly stable, where

$$A_{jk}(\lambda) = (1-\lambda)A(\bar{q}_j) + \lambda A(\bar{q}_k), \tag{33}$$

and $A(\bar{q}_j)$, $A(\bar{q}_k)$ are the vertex matrices of the system (11), corresponding to the vertices of the set (4).

According to Theorem 7, the standard system (11) with linear unity rank uncertainty structure is robustly stable if the edge systems (32), corresponding to all the edges of the value set (4) (not only exposed) are robustly stable. Asymptotic stability of the vertex systems (21) is also necessary for robust stability.

From Theorem 5 it follows that asymptotic stability of the vertex systems (21) is necessary and sufficient for robust stability of the positive system (11) with linear unity rank uncertainty structure.

In the case of the standard (or positive) system (11) with non-negative perturbation matrices the coefficients of the characteristic polynomial are polynomial functions of uncertain parameters, in general. In such a case:

- robust stability analysis of the standard system (11) is a very difficult problem (see [12–14,16], for example),
- robust stability of the positive system (11) is equivalent to the asymptotic stability of only one the positive system (31), according to Theorem 6.

From the above considerations it follows that robust stability conditions for the positive discrete-time systems with delays, given in this paper, are very simple in comparison with the suitable conditions for the standard systems.

5. Illustrative examples

Example 1. Check robust stability of the positive system (1) with $h = 2$, $m_k = 2$ for $k = 0,1,2$ and the matrices

$$A_{00} = \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0 \end{bmatrix},\ E_{01} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ E_{02} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \tag{34a}$$

$$A_{10} = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix},\ E_{11} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix},\ E_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{34b}$$

$$A_{20} = \begin{bmatrix} 0 & 0 \\ 0.2 & 0.1 \end{bmatrix},\ E_{21} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix},\ E_{22} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \tag{34c}$$

where $q = [q_0, q_1, q_2] \in Q$ with

$$Q = Q_0 \times Q_1 \times Q_2 \tag{35a}$$

and, for $k = 0,1,2$,

$$Q_k = \{[q_{k1}, q_{k2}] : q_{kr} \in [-0.1, 0.1], r = 1,2\} \tag{35b}$$

It is easy to check that the condition (5) holds. Hence, the system (1) with the matrices (34) is positive. Moreover, it is easy to see that this system has unity rank uncertainty structure (the conditions (7) holds) and the nominal system (8) is asymptotically stable. Hence, the assumptions 1 and 2 are satisfied. Therefore, we apply Theorem 5 to the robust stability analysis.

The set (35) of $m = m_0 + m_1 + m_2 = 6$ uncertain parameters has $L = 2^m = 2^6 = 64$ vertices. Hence, there is $L = 64$ the vertex system (21). Asymptotic stability of these systems is necessary and sufficient for robust stability of the system under consideration.

Computing the vertices of the set of uncertain parameters (35), the vertex matrices $A_l^v = A(\bar{q}_l)$, $l = 1, 2, ..., 64$, and coefficients of characteristic polynomials of the matrices $S(\bar{q}_l) = A_l^v - I_{\tilde{n}}$, $l = 1, 2, ..., 64$, we obtain that they are positive. Hence, the system is robustly stable, according to Theorem 5.

Example 2. Check robust stability of the positive system (1) with $h = 2$ and the matrices A_{00}, A_{10} and A_{20} of the forms given in (34) and

$$E_{01} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ E_{02} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ E_{11} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \tag{36a}$$

$$E_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \ E_{22} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{36b}$$

where the set Q is given by (35).

The system under consideration is a positive system with linear uncertainty structure with non-negative perturbation matrices because $E_{kr} \in \Re_+^{2\times 2}$ for $k = 0, 1, 2$, $r = 1, 2$. Therefore, we apply Theorem 6 to the robust stability analysis.

Computing the matrix $A^+ = A(q^+)$ from (29) with $q_{kr}^+ = 0.1$, $k = 0, 1, 2$, $r = 1, 2$, and characteristic polynomial of the matrix $S^+ = A^+ - I_6$, we obtain

$$\begin{aligned} &\det\left[(z+1)I_6 - A^+\right] \\ &= z^6 + 5.6z^5 + 12.5z^4 + 13.76z^3 + 7.24z^2 + 1.28z - 0.1. \end{aligned} \tag{37}$$

Because $\tilde{a}_0 = -0.1 < 0$, from Theorem 6 it follows that the system is not robustly stable.

6. Concluding remarks

Simple necessary and sufficient conditions for robust stability of the positive discrete-time linear system (1) with linear uncertainty structure in two cases: 1) unity rank uncertainty structure, 2) non-negative perturbation matrices, have been given.

It has been shown that:

- the positive system (1) with delays with linear unity rank uncertainty structure is robustly stable if and only if the positive systems (21) are asymptotically stable (Theorem 5),
- the positive system (1) with delays with non-negative perturbation matrices is robustly stable if and only if the system (31) is asymptotically stable (Theorem 6).

The proposed conditions for the positive systems have been compared with the suitable conditions for the standard systems.

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