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General Active-RC filter model for computer-aided design

S. KOZIEŁ[1,2] and S. SZCZEPAŃSKI[2*]

[1]Simulation Optimization Systems Research Laboratory, Department of Electrical and Computer Engineering, McMaster University, Hamilton, ON, L8S 4K1, Canada

[2]Faculty of Electronics, Telecommunications and Informatics, Gdańsk University of Technology, 11/12 Narutowicza St., Poland

Abstract. In the paper, a general topology of continuous-time Active-RC filter is presented. The model includes all possible Active-RC filter structures as particular cases and allows us to analyze them using a unified algebraic formalism. This makes it suitable for use in computer-aided analysis and design of Active-RC filters. By its construction, the model takes into account the finite DC gain and the finite bandwidth as well as non-zero output resistance of operational amplifiers. Filters with ideal OPAMPs can be treated as particular cases. Sensitivity and noise analysis of Active-RC filters is also performed in the proposed general setting. The correctness of the model is verified by comparison with SPICE simulation.

Key words: Active-RC filters, continuous-time filters, sensitivity, noise analysis, computer-aided design.

1. Introduction

Continuous-time analog filters based on operational amplifiers (OPAMPs) and RC elements (Active-RC filters) provide solutions for various signal-processing tasks. Many synthesis and design methods for different types and architectures of this class of filters have been reported [1–3]. Performance demands for continuous-time Active-RC filters have increased significantly during the last several years to meet the needs of rapidly developing applications, such as ADSL [4], VDSL [5,6] WCDMA [7,8] RFIC receivers for PANs [9], GSM baseband transmitters [10]; recently, Active-RC filter for frequencies beyond 300 MHz has been successfully implemented [11]. This creates the need for developing new techniques, especially computer-aided design tools, concerning both transistor- and system-level design and optimization. The key factor in this development is creating good models of circuits and systems, in this case, Active-RC filters, that would be accurate, general enough to include as many of particular cases as possible and easy to use in automated design/optimization systems.

In this paper, we consider a general Active-RC filter topology suitable for computer-aided analysis, design and optimization of Active-RC filters. The presented model includes all possible structures of filters of this class. It utilizes matrix formalism for circuit description that makes all the resulting formulas easy to implement in computer software. Due to the limited space, in this paper we can merely give an outline of the model and carry out its verification using suitable filter examples. Applications for computer-aided design and optimization will be covered in a separate work. In [12], interested readers can find details concerning analogous approach to OTA-C filters.

The paper is organized as follows. In Section 2, we present a general Active-RC filter topology and develop a matrix description of this structure. In its basic formulation, the model takes into consideration finite gain and bandwidth as well as non-zero output resistance of filter OPAMPs. Active-RC filters with ideal OPAMPs are treated as special cases. In Section 3 we derive explicit formulas for sensitivity functions for arbitrary filter of the considered class. Section 4 deals with noise analysis in general setting. In Section 5 we verify the proposed model by comparing theoretical results with SPICE simulation on transistor-level using chosen benchmark circuits. Section 6 concludes the paper.

2. Active-RC filter model

Figure 1 shows the general topology of Active-RC filter. The structure contains $n+1$ nodes denoted as $x_i,\ i=0,...,n$, passive network consisting of admittances $y_{bi},\ i=1,...,n$ and $y_{ij},\ i,j=1,...,n$ as well as k operational amplifiers $O_i,\ i=1,...,k$. We denote the input node as x_0 and the output node as x_n. We also denote the ground node as x_g for the convenience of further description. We will also use symbols x_i to denote voltages at the respective nodes (voltage corresponding to x_g is 0 by definition). The input and output nodes of OPAMPs are connected to internal nodes so that $O_{i+}, O_{i-} \in \{x_0, x_1, ..., x_n, x_g\}$ and $O_{io} \in \{x_1, x_2, ..., x_n\}$ (we exclude the situation when the output of the amplifier is connected to the input or ground node for obvious reasons). Some additional and obvious constraints apply, which exclude impractical circuits, such as $O_{i+} \neq O_{i-}$ (inputs of the OPAMP cannot be short-connected) or $O_{io} \neq O_{jo}$ for $i \neq j$ (the outputs of two different amplifiers cannot be connected to the same node). We assume, without loss of generality, that any admittance of the circuit may be, in general, a parallel connection of resistor and capacitor, as shown in Fig. 2. It is clear that any conceivable Active-RC filter is a particular case of the structure in Fig. 1.

*e-mail: stanisla@pg.gda.pl

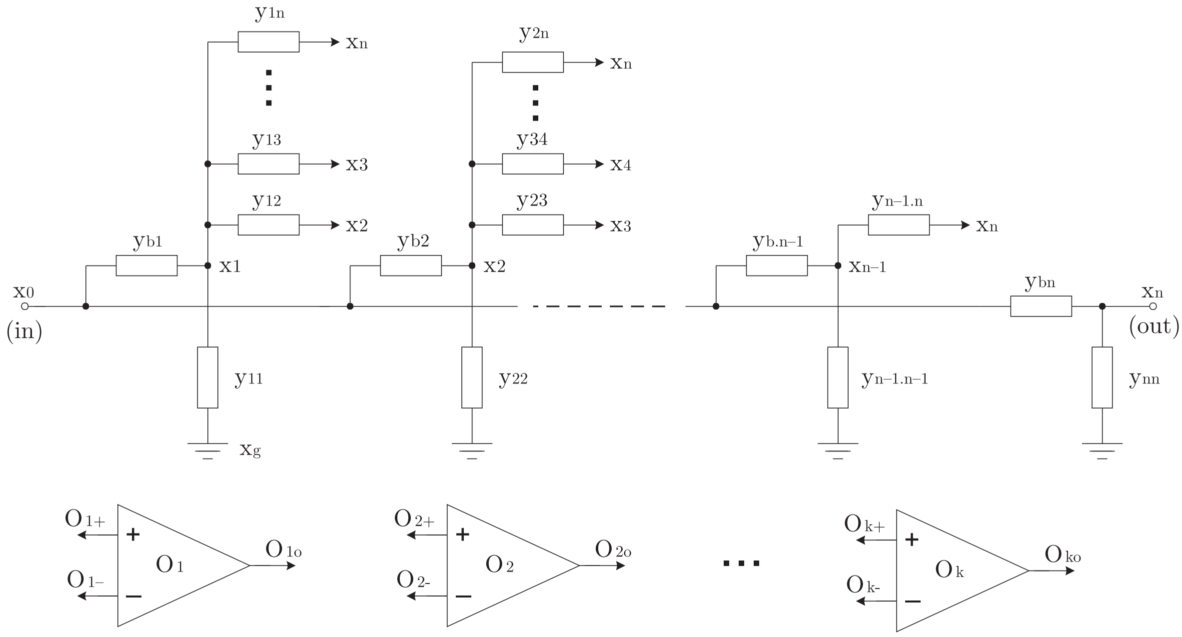

Fig. 1. General topology of Active-RC filter

Fig. 2. General form of admittance element of filter in Fig. 1 (here for y_{ij})

As a first step, we shall perform an analysis of the circuit in Fig. 1 that aims at developing compact formulas for evaluating filter transfer function. These formulas have to be general in order to be applicable to any particular case of the filter structure in Fig. 1 assuming either ideal OPAMPs or OPAMPs with finite gain and output resistance. In the sequel, additional non-ideal effects will be considered such as electric noise generated by filter resistors and OPAMPs. Since the presented model is intended to use primarily in a computer-aided design and optimization systems, we shall utilize algebraic description so that resulting formulas are easily implemented in computer software. For the purpose of the analysis we shall use the OPAMP model shown in Fig. 3, where A is an open-loop amplifier gain and r_o is its output resistance. Because we use classical nodal analysis, more convenient is equivalent model shown in Fig. 4, where $g_o = 1/r_o$ and $g_m = A/r_o = Ag_o$. In order to distinguish model parameters for different OPAMPs we use the symbols A_i, g_{oi} and g_{mi} to denote open loop gain, output conductance and transconductance of the amplifier $O_i, i = 1, ..., k$.

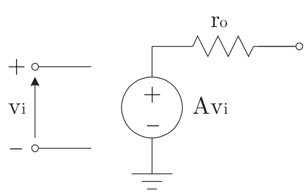

Fig. 3. Operational amplifier model for analysis of the filter in Fig. 1

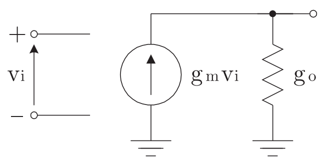

Fig. 4. Equivalent current-source operational amplifier model for analysis of the filter in Fig. 1

The circuit in Fig. 1 can be described by the following linear system

$$(\boldsymbol{Y} + \boldsymbol{G})\,\boldsymbol{x} = (\boldsymbol{B} + \boldsymbol{G}_0)\,u_{in} \tag{1}$$

where $\boldsymbol{x}$ is vector of node voltages (here $x_n = u_{out}$ – output voltage; these symbols will be used interchangeably)

$$\boldsymbol{x} = \left[\, x_1 \; x_2 \; \cdots \; x_n \,\right]^{T^T} \tag{2}$$

and

$$\boldsymbol{B} = \left[\, y_{b1} \; y_{b2} \; \cdots \; y_{bn} \,\right]^T \tag{3}$$

$$\boldsymbol{Y} = \begin{bmatrix} y_{b1} + \sum_{j=1}^{n} y_{1j} & -y_{12} & \cdots & -y_{1n} \\ -y_{12} & y_{b2} + \sum_{j=1}^{n} y_{2j} & & -y_{2n} \\ \vdots & & \ddots & \vdots \\ -y_{1n} & -y_{2n} & \cdots & y_{bn} + \sum_{j=1}^{n} y_{nj} \end{bmatrix} \tag{4}$$

$$\boldsymbol{G} = [g_{ij}]_{i,j=0}^{n}\,,$$

$$g_{ij} = \begin{cases} -g_{mq} & \text{if } O_{qo} = x_i,\ O_{q+} = x_j \text{ for some } q \in \{1, ..., k\} \\ g_{mq} & \text{if } O_{qo} = x_i,\ O_{q-} = x_j \text{ for some } q \in \{1, ..., k\} \\ g_{oq} & \text{if } O_{qo} = x_i \text{ for some } q \in \{1, ..., k\} \text{ and } i = j \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$\boldsymbol{G}_0 = \begin{bmatrix} g_{01} \\ \vdots \\ g_{0n} \end{bmatrix},$$

$$g_{0i} = \begin{cases} g_{mq} & \text{if } O_{qo} = x_i,\ O_{q+} = x_0 \text{ for some } q \in \{1,...,k\} \\ -g_{mq} & \text{if } O_{qo} = x_i,\ O_{q-} = x_0 \text{ for some } q \in \{1,...,k\} \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Using (1)–(6) one can easily calculate the vector $\boldsymbol{x}$ of internal node voltages of the circuit in Fig. 1:

$$\boldsymbol{x} = [\boldsymbol{Y} + \boldsymbol{G}]^{-1} (\boldsymbol{B} + \boldsymbol{G}_0)\, u_{in} \tag{7}$$

and its transfer function $H_{A,r_o}(s)$

$$H_{A,r_o}(s) = \frac{u_{out}}{u_{in}} = \boldsymbol{C}\, [\boldsymbol{Y} + \boldsymbol{G}]^{-1} (\boldsymbol{B} + \boldsymbol{G}_0) \tag{8}$$

where $\boldsymbol{C}$ is $1 \times n$ matrix defined as

$$\boldsymbol{C} = \begin{bmatrix} 0 \cdots 0\ 1 \end{bmatrix}. \tag{9}$$

In practice, Active-RC filters are mostly implemented in fully differential structures. Due to this we may assume that element values (e.g. R_{ij}, C_{ij}) can take both positive and negative values, which can be accomplished by cross-coupling of corresponding physical elements. More specifically, if the element, say R_{ij}, is cross-coupled (i.e. put between positive (negative) output of an amplifier and positive [negative] input of another one, see e.g. resistor R_1 in Fig. 5), this reflects in Eq. (1) so that the appropriate term has the form $g_{ij}(x_i - (-x_j)) = g_{ij}x_i - (-g_{ij})x_j$, i.e. the original '–' from node voltage is moved into filter element, here $g_{ij} = 1/R_{ij}$, but only while considering non-diagonal elements of matrix $\boldsymbol{Y}$. Obviously, the value of physical element remains positive. Negative value of the corresponding matrix entry is equivalent to cross-coupling. In case of single-ended implementation, negative elements can be realized using inverters.

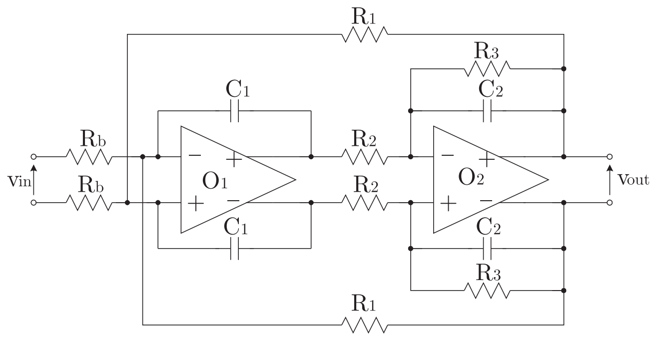

Fig. 5. Fully differential second-order Active-RC filter

Note that Eq. (8) gives the filter transfer function assuming finite gain and non-zero output resistance of its operational amplifiers. In order to determine the transfer function assuming ideal amplifiers, we have to perform some additional operations. As a first step we consider the case when output resistances of filter operational amplifiers are neglected. From the point of view of the OPAMP model in Fig. 4 it is equivalent to g_m and g_o going to infinity. As a result, if the output of one of the filter OPAMPs, say O_q, is connected to some internal node, say x_i, then it is seen, after dividing the corresponding (i-th) equation of the system (1) by g_{oq}, that all the factors containing the admittances y_{ib} and y_{ij}, $j = 1, ..., n$, become negligible, and the whole equation becomes just the relation between differential input voltage of the OPAMP and its output voltage. In terms of algebraic description of the filter in Fig. 1 it is expressed by the following matrix operations

$$\boldsymbol{Y} \to \boldsymbol{P}_Y\boldsymbol{Y},\ \boldsymbol{B} \to \boldsymbol{P}_Y\boldsymbol{B},\ \boldsymbol{G} \to \boldsymbol{P}_G\boldsymbol{G},\ \boldsymbol{G}_0 \to \boldsymbol{P}_G\boldsymbol{G}_0 \tag{10}$$

where

$$\boldsymbol{P}_Y = \begin{bmatrix} p_{Y1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{Yn} \end{bmatrix}, \quad p_{Yi} = \begin{cases} 0 & O_{oq} = x_i \text{ for some } q \in \{1,...,k\} \\ 1 & \text{otherwise} \end{cases} \tag{11}$$

$$\boldsymbol{P}_G = \begin{bmatrix} p_{G1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{Gn} \end{bmatrix}, \quad p_{Gi} = \begin{cases} 1/g_{oq} & O_{oq} = x_i \text{ for some } q \in \{1,...,k\} \\ 0 & \text{otherwise} \end{cases}. \tag{12}$$

The modified equations for the filter in Fig. 1 with zero output resistance of OPAMPs take the following form

$$(\boldsymbol{P}_Y\boldsymbol{Y} + \boldsymbol{P}_G\boldsymbol{G})\,\boldsymbol{x} = (\boldsymbol{P}_Y\boldsymbol{B} + \boldsymbol{P}_G\boldsymbol{G}_0)\, u_{in} \tag{13}$$

and the modified transfer function formula is

$$H_A(s) = \boldsymbol{C}\, [\boldsymbol{P}_Y\boldsymbol{Y} + \boldsymbol{P}_G\boldsymbol{G}]^{-1} (\boldsymbol{P}_Y\boldsymbol{B} + \boldsymbol{P}_G\boldsymbol{G}_0). \tag{14}$$

As the next step we consider the case when filter OPAMPs is ideal, that is not only output resistance is zero but also open-loop gain A is infinite. In this case the differential input voltage of the amplifier becomes zero, which means that some of the node voltages have to be identified with each other (those corresponding to positive and negative input of the same amplifier) and some set to zero (those corresponding to either positive or negative input of the amplifier whose second input is connected to ground). This means that the number of equations in (1) as well as the number of unknown voltages has to be reduced by the number k of operational amplifiers in the filter. In terms of algebraic description of the filter structure in Fig. 1 it is accomplished by the following matrix operations

$$\begin{aligned} &\boldsymbol{P}_Y\boldsymbol{Y} \to \boldsymbol{L}\boldsymbol{P}_Y\boldsymbol{Y}\boldsymbol{R}, \quad \boldsymbol{P}_Y\boldsymbol{B} \to \boldsymbol{P}_Y\boldsymbol{B} - \boldsymbol{L}\boldsymbol{P}_Y\boldsymbol{Y}\boldsymbol{R}_B, \\ &\boldsymbol{P}_G\boldsymbol{G} \to 0, \quad \boldsymbol{P}_G\boldsymbol{G}_0 \to 0, \end{aligned} \tag{15}$$

where

$$\boldsymbol{L} = [l_{ij}]_{i=1,...,n-k;\ j=1,...,n} \tag{16}$$

with $l_{ij} = 1$ if x_j is the i-th (counting from x_1) node with no OPAMP output connected to it and 0 otherwise,

$$\boldsymbol{R} = [r_{ij}]_{i=1,...,n;\ j=1,...,n-k} \tag{17}$$

which is constructed from $n \times n$ identity matrix $\boldsymbol{I}_n$ in such a way that: (i) if $O_{q+} = x_i$ and $O_{q-} = x_g$ (or $O_{q+} = x_g$ and $O_{q-} = x_i$), i.e. node x_i is connected to the input of some OPAMP and the other input of the same OPAMP is connected to ground or input node then the i-th column of $\boldsymbol{I}_n$ is removed;

(ii) if $O_{q+} = x_i$ and $O_{q-} = x_j$ (or $O_{q+} = x_j$ and $O_{q-} = x_i$), i.e. two nodes x_i and x_j are connected to two inputs of the same OPAMP then i-th and j-th column of $\boldsymbol{I}_n$ are concatenated. The columns of $\boldsymbol{R}$ are arranged in such a way that $r_{ij} = 0$ for $j > i$, $i, j = 0, ..., n-k$. Finally, matrix $\boldsymbol{R}_B$ is defined as

$$\boldsymbol{R}_B = \begin{bmatrix} r_{b.1} & \cdots & r_{b.n} \end{bmatrix}^T \tag{18}$$

where $r_{b.i} = 1$ if $O_{q+} = x_i$ and $O_{q-} = x_0$ (or $O_{q+} = x_0$ and $O_{q-} = x_i$), i.e. input node (x_0) is connected to the input of some OPAMP so that the other input is connected to x_i, and $r_{b.i} = 0$ otherwise.

Note also that above operations lead to identification of some node voltages and removing others from the equations. As a result we have a new vector $\bar{\boldsymbol{x}} = [\bar{x}_1 \ \bar{x}_2 \ \ldots \ \bar{x}_{n-k}]^T$ of unknown voltages, which is in the following relation with the original vector $\boldsymbol{x}$

$$\bar{\boldsymbol{x}} = \bar{\boldsymbol{R}}\boldsymbol{x} \tag{19}$$

where the matrix $\bar{\boldsymbol{R}}$ is constructed from the matrix $\boldsymbol{R}^T$ so that each row of $\bar{\boldsymbol{R}}$ contains exactly one 1 (if there were two or more 1-s in the corresponding column of $\boldsymbol{R}$ then the one with higher row index is removed). In other words, the vector $\bar{\boldsymbol{x}}$ refers to those nodes which either have no OPAMP input connected to them or, if the node has the input of some OPAMP connected to it, the second input of the same OPAMP is connected to another (but not ground) node of the filter. The last component, i.e. x_{n-k} corresponds to the output voltage v_{out} of the filter. The modified equations for the filter in Fig. 1 with ideal OPAMPs take the following form

$$\bar{\boldsymbol{Y}}\bar{\boldsymbol{x}} = \bar{\boldsymbol{B}}u_{in} \tag{20}$$

where

$$\bar{\boldsymbol{Y}} = \boldsymbol{L}\boldsymbol{P}_Y\boldsymbol{Y}\boldsymbol{R}, \quad \bar{\boldsymbol{B}} = \boldsymbol{L}\boldsymbol{P}_Y\boldsymbol{B} - \boldsymbol{L}\boldsymbol{P}_Y\boldsymbol{Y}\boldsymbol{R}_B. \tag{21}$$

In order to calculate transfer function of the filter in Fig. 1 we have to solve the modified system (1) with respect to $\bar{\boldsymbol{x}}$. The modified transfer function formula is

$$H(s) = \bar{\boldsymbol{C}}\bar{\boldsymbol{Y}}^{-1}\bar{\boldsymbol{B}} \tag{22}$$

where $\bar{\boldsymbol{C}}$ is $1 \times n-k$ matrix defined as

$$\bar{\boldsymbol{C}} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{23}$$

An immediate consequence of Eq. (27) is that the maximum transfer function order of the filter with n nodes and k OPAMPs is $n-k$, since this is the size of the matrix $\widetilde{\boldsymbol{C}}$.

Formulas (8), (14) and (22) allow us to calculate transfer function of any particular case of the filter in Fig. 1 assuming either non-ideal or ideal OPAMPs. OPAMP gains implicitly present in matrices $\boldsymbol{G}$ and $\boldsymbol{G}_0$ (Eqs (5) and (6)) may be frequency dependent.

It is important to note that the matrices $\boldsymbol{Y}$, $\boldsymbol{G}$ and $\boldsymbol{B}$, describing the filter can be written down by circuit diagram inspection. Matrices $\boldsymbol{P}_Y$, $\boldsymbol{P}_G$, $\boldsymbol{L}$ and $\boldsymbol{R}$ are in one-to-one relation with matrix $\boldsymbol{G}$ and can be easily constructed from it. On the other hand, there is exactly one Active-RC filter corresponding to the given set of matrices $\boldsymbol{Y}$, $\boldsymbol{G}$ and $\boldsymbol{B}$ of the form (3)–(6). This allows us move the problem of Active-RC filter analysis, design and optimization into algebraic domain, which can be easily handled by computer.

Note that if no filter OPAMP has its inputs connected to the input node then the matrices $\boldsymbol{G}_0$ and $\boldsymbol{R}_B$ are zero and most of the previous formulas take simpler form.

For the sake of illustration consider some examples of Active-RC filters. Let us start with a second-order low-pass filter shown in Fig. 5 (actually, the figure shows its fully differential version).

The matrices $\boldsymbol{Y}$, $\boldsymbol{G}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ corresponding to this filter are (we assume both OPAMPs to be identical):

$$\boldsymbol{Y} = \begin{bmatrix} R_b^{-1} + R_1^{-1} + sC_1 & -sC_1 & 0 & R_1^{-1} \\ -sC_1 & R_2^{-1} + sC_1 & -R_2^{-1} & 0 \\ 0 & -R_2^{-1} & R_2^{-1} + R_3^{-1} + sC_2 & -R_3^{-1} - sC_2 \\ R_1^{-1} & 0 & -R_3^{-1} - sC_2 & R_1^{-1} + R_3^{-1} + sC_2 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ Ag_o & g_o & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Ag_o & g_o \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} R_b^{-1} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \ \boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

Remaining matrices are:

$$\boldsymbol{P}_Y = \mathrm{diag}\{1, 0, 1, 0\},$$
$$\boldsymbol{P}_G = \mathrm{diag}\{0, 1/g_o, 0, 1/g_o\}, \ \ \bar{\boldsymbol{C}} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \tag{25}$$

Evaluation of formulas (22) and (14) (we put $R_b = R_1 = R_2 = R_3 = R$ for simplicity) gives:

$$H(s) = \frac{\frac{1}{R^2C_1C_2}}{s^2 + \frac{s}{RC_2} + \frac{1}{R^2C_1C_2}} \tag{26}$$

$$H_A(s) = \frac{\frac{1}{R^2C_1C_2(1+A^{-1})^2}}{s^2 + s\left(\frac{(1+2A^{-1})}{RC_2(1+A^{-1})} + \frac{1}{RC_1(1+A)}\right) + \frac{1+3/(1+A)^2}{R^2C_1C_2}} \tag{27}$$

We omitted $H_{A,r_o}(s)$ (formula (8)) because it is quite long and does not give additional insight at this point. If we assume now that $A = A(s) = A_0/(1 + s/\omega_0)$, where ω_0 is 3dB bandwidth of open-loop OPAMP, we can easily calculate filter transfer function distortion due to finite OPAMP gain-bandwidth product. The above results are given just for illustrative purposes in practice they are obtained and processed by computer software implementing the presented filter model.

As a second example consider a well-known Sallen-Key biquad [2] shown in Fig. 6. The matrices $\boldsymbol{Y}$, $\boldsymbol{G}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ corresponding to this filter are:

$$\boldsymbol{Y} = \begin{bmatrix} R_1^{-1} + R_2^{-1} + sC_1 & -R_2^{-1} & 0 & -sC_1 \\ -R_2^{-1} & R_2^{-1} + sC_2 & 0 & 0 \\ 0 & 0 & R_a^{-1} + R_b^{-1} & -R_b^{-1} \\ -sC_1 & 0 & -R_b^{-1} & R_b^{-1} + sC_1 \end{bmatrix}$$

$$\boldsymbol{G}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&-Ag_o&Ag_o&g_o\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}R_1^{-1}\\0\\0\\0\end{bmatrix}^T,\ \boldsymbol{C}=\begin{bmatrix}0&0&0&1\end{bmatrix} \tag{28}$$

Remaining matrices are:

$$\boldsymbol{P}_Y=\mathrm{diag}\{1,1,1,0\},$$
$$\boldsymbol{P}_G=\mathrm{diag}\{0,0,0,1/g_o\},\quad \bar{\boldsymbol{C}}=\begin{bmatrix}0&0&1\end{bmatrix}$$
$$\boldsymbol{L}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix},\quad \boldsymbol{R}=\begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&0&1\end{bmatrix}^T \tag{29}$$

Evaluation of formulas (22) and (14) (we put $R_1=R_2=R$, $C_1=C_2=C$ and $K=1+R_b/R_a$ for simplicity) gives:

$$H(s)=\frac{\frac{K}{R^2C^2}}{s^2+s\frac{3-K}{RC}+\frac{1}{R^2C^2}} \tag{30}$$

$$H_A(s)=\frac{\frac{K}{R^2C^2}}{s^2\left(1+\frac{K}{A}\right)+s\frac{3-K+3K/A}{RC}+\frac{1+K/A}{R^2C_1C_2}}. \tag{31}$$

As before we omitted $H_{A,r_o}(s)$.

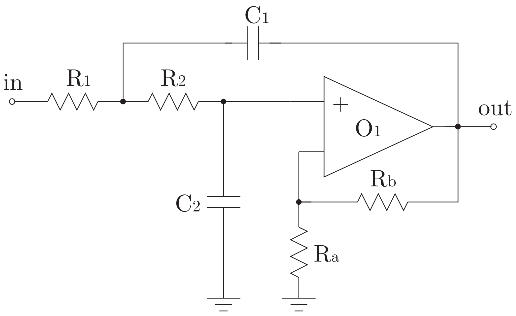

Fig. 6. Circuit diagram of the Salley-Key low-pass biquad

3. Sensitivity analysis

Using the general Active-RC filter model described in the previous section, it is possible to easily calculate sensitivity functions of any filter of this class. In this section we focus on first-order sensitivity [1] of filter transfer function. By definition, sensitivity function with respect to any filter element z is calculated from the following formula:

$$S_z^{H(s)}=\frac{z}{H(s)}\frac{\partial H(s)}{\partial z}=\frac{z}{H(s)}\lim_{h\to 0}\frac{H(s,z+h)-H(s,z)}{h} \tag{32}$$

where in the last term of (32) there is explicitly written functional dependence of H on z. Matrix description of the filter in Fig. 1 makes calculation of sensitivity functions extremely easy and convenient. For the sake of example let us show how to calculate sensitivity functions of the filter in Fig. 1 assuming ideal OPAMPs. Suppose that we want to calculate first-order sensitivity of $H(s)$ with respect to grounded admittance $y_{ii}, i=1,...,n$, i.e.

$$\begin{aligned}S_{y_{ii}}^{H(s)}&=\frac{y_{ii}}{H(s)}\frac{\partial H(s)}{\partial y_{ii}}\\&=\frac{y_{ii}}{H(s)}\lim_{h\to 0}\frac{H(s,y_{ii}+h)-H(s,y_{ii})}{h}\end{aligned} \tag{33}$$

Now, we have

$$H(s,y_{ii}+h)=\bar{\boldsymbol{C}}\left[\boldsymbol{L}\boldsymbol{P}_Y\left(\boldsymbol{Y}+h\boldsymbol{E}_{ii}\right)\boldsymbol{R}\right]^{-1}\boldsymbol{L}\boldsymbol{P}_Y\boldsymbol{B} \tag{34}$$

where $\boldsymbol{E}_{ii}$ is $n\times n$ elementary matrix (all zeros except 1 in position ii). Simple calculations yield:

$$S_{y_{ii}}^{H(s)}=-\frac{y_{ii}}{H(s)}\boldsymbol{S}_L\boldsymbol{E}_{ii}\boldsymbol{S}_R \tag{35}$$

where $\boldsymbol{S}_L=\bar{\boldsymbol{C}}\bar{\boldsymbol{Y}}^{-1}\boldsymbol{L}\boldsymbol{P}_Y$ is $1\times n$ row vector, and $\boldsymbol{S}_R=\boldsymbol{R}\bar{\boldsymbol{Y}}^{-1}\bar{\boldsymbol{B}}$ is $n\times 1$ column vector, with $\bar{\boldsymbol{Y}}$, $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{C}}$ defined by (21) and (23), respectively. Corresponding results for floating ($y_{ij},\ i,j=1,...,n,\ i\neg j$) and input ($y_{bi},\ i=1,...,n$) admittances are:

$$S_{y_{ij}}^{H(s)}=\frac{y_{ii}}{H(s)}\boldsymbol{S}_L\left(\boldsymbol{E}_{ij}+\boldsymbol{E}_{ji}-\boldsymbol{E}_{ii}-\boldsymbol{E}_{ij}\right)\boldsymbol{S}_R \tag{36}$$

$$S_{y_{bi}}^{H(s)}=-\frac{y_{bi}}{H(s)}\boldsymbol{S}_L\left[\boldsymbol{E}_{ii}\boldsymbol{S}_R-\boldsymbol{e}_i\right] \tag{37}$$

where $\boldsymbol{e}_i$ is $n\times 1$ elementary vector (all 0 except 1 in position i). Note that in formulas (35)–(37) we have three common terms: $H(s)$, $\boldsymbol{S}_L$, and $\boldsymbol{S}_R$. All these components have to be calculated only once in order to get all sensitivity functions of the filter.

Having sensitivity functions with respect to y_{ij} and y_{bi} one can easily calculate sensitivity with respect to individual filter elements ($R_{ij},\ C_{ij},\ R_{bi},\ C_{bi}$):

$$\begin{aligned}S_{R_{ij}}^{H(s)}&=\frac{R_{ij}}{H(s)}\frac{\partial H(s)}{\partial R_{ij}}=\frac{R_{ij}}{y_{ij}}\frac{y_{ij}}{H(s)}\frac{\partial H(s)}{\partial y_{ij}}\frac{\partial y_{ij}}{\partial R_{ij}}\\&=-\frac{1}{R_{ij}y_{ij}}S_{y_{ij}}^{H(s)}\end{aligned} \tag{38}$$

$$\begin{aligned}S_{C_{ij}}^{H(s)}&=\frac{C_{ij}}{H(s)}\frac{\partial H(s)}{\partial C_{ij}}=\frac{C_{ij}}{y_{ij}}\frac{y_{ij}}{H(s)}\frac{\partial H(s)}{\partial y_{ij}}\frac{\partial y_{ij}}{\partial C_{ij}}\\&=\frac{sC_{ij}}{y_{ij}}S_{y_{ij}}^{H(s)}\end{aligned} \tag{39}$$

$$\begin{aligned}S_{R_{bi}}^{H(s)}&=\frac{R_{bi}}{H(s)}\frac{\partial H(s)}{\partial R_{bi}}=\frac{R_{bi}}{y_{bi}}\frac{y_{bi}}{H(s)}\frac{\partial H(s)}{\partial y_{bi}}\frac{\partial y_{bi}}{\partial R_{bi}}\\&=-\frac{1}{R_{bi}y_{bi}}S_{y_{bi}}^{H(s)}\end{aligned} \tag{40}$$

$$\begin{aligned}S_{C_{bi}}^{H(s)}&=\frac{C_{bi}}{H(s)}\frac{\partial H(s)}{\partial C_{bi}}=\frac{C_{bi}}{y_{bi}}\frac{y_{bi}}{H(s)}\frac{\partial H(s)}{\partial y_{bi}}\frac{\partial y_{bi}}{\partial C_{bi}}\\&=\frac{sC_{bi}}{y_{bi}}S_{y_{bi}}^{H(s)}.\end{aligned} \tag{41}$$

For the sake of illustration let us calculate sensitivity functions for the second-order filter in Fig. 5 assuming $R_b=R_1=R_2=R_3=R$. Transfer function of this filter is given by (46) and we will denote its denominator as D(s). Matrices $\boldsymbol{S}_L$ and $\boldsymbol{S}_R$ in this case are:

$$\boldsymbol{S}_L=\frac{1}{RC_1C_2D(s)}\begin{bmatrix}1&0&-sRC_1&0\end{bmatrix} \tag{42}$$

$$\boldsymbol{S}_R = \frac{1}{R^2 C_1 C_2 D(s)} \left[0 \; -1 - sRC_2 \; 0 \; 1 \right]^T . \qquad (43)$$

Using (62) and (63) we can calculate sensitivity functions with respect to all filter elements:

$$\begin{aligned} S_{R_b}^{H(s)} &= \tfrac{1}{\bar{D}(s)}, & S_{R_1}^{H(s)} &= -\tfrac{1}{\bar{D}(s)}, \\ S_{R_2}^{H(s)} &= \tfrac{sRC_1(1+sRC_2)}{\bar{D}(s)}, & S_{R_3}^{H(s)} &= -\tfrac{sRC_1}{\bar{D}(s)}, \\ S_{C_1}^{H(s)} &= \tfrac{1+sRC_2}{\bar{D}(s)}, & S_{C_2}^{H(s)} &= \tfrac{sRC_1}{\bar{D}(s)} \end{aligned} \qquad (44)$$

where $\bar{D}(s) = R^2 C_1 C_2 D(s)$.

4. Noise analysis

Noise performance is one of the very important characteristics of Active-RC filters. A number of papers have been published dealing with noise in Active-RC filters, mostly some particular topologies such as single amplifier filters, state-space filters (eg. [13–15]) but also in general setting [16]. In this section, we present a procedure for evaluating noise in Active-RC filters, which is based on the matrix description of the general Active-RC filter model presented in Section 2.

We shall carry out the noise analysis of the filter structure in Fig. 1. The output noise of any Active-RC filter is a combination of the noise contribution of its all operational amplifiers and resistors (we treat filter capacitors as noiseless). The noise of operational amplifier can be described by the equivalent input referred noise voltage source vn as shown in Fig. 7. Spectral density $S_n(f)$ of the noise source can be modelled as:

$$S_n(f) = S_t + \frac{S_f}{f} \qquad (45)$$

where both S_t (thermal noise component) and S_f (flicker noise component) depend on amplifier topology, however, in general we do not need to restrict ourselves to this model. Noise of the resistor of value R will be represented by the corresponding spectral density $4kTR$, where k is Boltzmann's constant, and T is absolute temperature. We shall assume that noise sources associated to different OPAMPs and resistors are statistically independent.

Our immediate goal is to obtain the explicit formula for output (and/or input) noise spectrum of the general Active-RC filter in Fig. 1. In order to do this, one has to consider what is the contribution of the noise of each individual operational amplifier and resistor to the output noise spectrum of the filter.

We start from the noise contribution of filter OPAMPs $O_q, q = 1, ..., k$. Let v_{Oi} denote the input referred noise voltage of the amplifier O_q, whose spectral density is $S_{Oq}(f)$. We assume that v_{Oq} includes the noise of output resistance of the amplifier. Figure 8 shows the model of OPAMP including noise source. Suppose that we have $O_{q+} = x_i$, $O_{q-} = x_j$, and $O_{qo} = x_k$ (recall that in general O_{q+} or O_{q-} may be equal to x_0 or x_g as well). The k-th equation of system (1) contains the factor $(x_j - x_i)g_{mq}$, which, in presence of noise source has to be replaced by the following: $(x_j - x_i - v_{Oq})g_{mq}$. As a next step, we move the factor $-v_{Oq}g_{mq}$ into the right-hand side of the equation and solve the whole system in order to find the output voltage of the filter due to v_{Oq}.

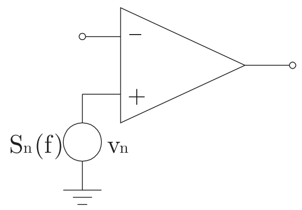

Fig. 7. Equivalent input voltage noise source representation of noise in OPAMP

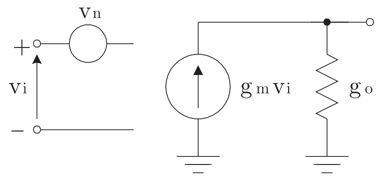

Fig. 8. Operational amplifier model with input noise voltage source

Let us denote by $\boldsymbol{H}_{cv}$ the $1 \times n$ vector defined as follows

$$\boldsymbol{H}_{cv}(s) = \left[H_1(s) \cdots H_n(s) \right] = \boldsymbol{C}(\boldsymbol{Y} + \boldsymbol{G})^{-1} . \qquad (46)$$

Note that component H_i of the vector $\boldsymbol{H}_{cv}$ can be interpreted as current-to-voltage transfer function from i-th node of the filter to its output. Now, we can calculate output noise U_{Oq} of the filter due to amplifier O_q as follows

$$U_{Oq} = H_{Oqo} g_{mq} v_{Oq}. \qquad (47)$$

Corresponding spectral density $S_{o.Oq}(f)$ is given by the formula:

$$S_{o.Oq}(f) = S_{Oq}(f) \left| H_{Oqo}(j2\pi f) \right|^2 g_{mq}^2 \qquad (48)$$

The total output noise voltage spectrum $S_O(f)$ of the filter in Fig. 1 due to its operational amplifiers can be now calculated using (48) and the assumption of statistical independence of noise sources as

$$S_O(f) = \sum_{q=1}^{k} S_{Oq}(f) \left| H_{Oqo}(j2\pi f) \right|^2 g_{mq}^2. \qquad (49)$$

As a next step, we shall calculate the output noise of the filter in Fig. 1 due to its resistors. First, we consider input resistors. Denote by v_{bi} the noise voltage of resistor R_{bi} (corresponding spectral density is $4kTR_{bi}$). Recall that R_{bi} is the part of input admittance y_{bi} as shown in Fig. 2. Output noise voltage U_{bi} due to this resistor can be calculated from the following equation

$$U_{bi} = H_i g_{bi} v_{bi} \qquad (50)$$

which is because v_{bi} acts in this case as an input voltage of the filter. Equation (50) can be rewritten in matrix form as follows:

$$U_{bi} = \boldsymbol{H}_{cv} \boldsymbol{e}_i g_{bi} v_{bi} \qquad (51)$$

where $\boldsymbol{e}_i$ is usual $n \times 1$ elementary vector. Corresponding spectral density $S_{bi}(f)$ is given by the formula:

$$S_{bi}(f) = 4kTR_{bi} |H_i(2\pi f)|^2 g_{bi}^2 = 4kT|g_{bi}| \cdot |H_i(2\pi f)|^2 \qquad (52)$$

or, in matrix form

$$S_{bi}(f) = 4kT|g_{bi}| \cdot |\boldsymbol{H}_{cv}(2\pi f)\boldsymbol{e}_i|^2. \qquad (53)$$

The noise due to feedback resistors $R_{ij},\ i,j = 1,...,n$ can be calculated in a similar way (recall that R_{ij} is part of admittance y_{ij}). Denote by v_{kl} the noise voltage of resistor R_{ij} (corresponding spectral density is $4kTR_{ij}$). We consider two cases. If $i = j$, i.e. R_{ij} is grounded resistor, then the i-th equation of system (1) contains the factor $x_i g_{ii}$, which, in presence of noise source has to be replaced by the following: $(x_i - v_i)g_{ii}$. As a next step, we move the factor $-v_{ii}g_{ii}$ into the right-hand side of the equation and solve the whole system in order to find the output voltage of the filter due to v_{ii}. Output noise voltage U_{ii} due to resistor R_{ii} can be then calculated from the following equation

$$U_{ii} = H_i g_{ii} v_{ii} \tag{54}$$

Corresponding spectral density $S_{ii}(f)$ is given by the formula:

$$S_{ii}(f) = 4kTR_{ii}|H_i(2\pi f)|^2 g_{ii}^2 = 4kT|g_{ii}| \cdot |H_i(2\pi f)|^2. \tag{55}$$

We can also rewrite (54) and (55) in matrix form, which gives

$$U_{ii} = \boldsymbol{H}_{cv}\boldsymbol{e}_i g_{ii} v_{ii} \tag{56}$$

$$S_{ii}(f) = 4kT|g_{ii}| \cdot |\boldsymbol{H}_{cv}(2\pi f)\,\boldsymbol{e}_i|^2. \tag{57}$$

The second case refers to floating resistors, i.e. when $i \neq j$. In this case we have to modify both i-th and j-th equation of the system (1): in i-th equation the term $(x_i - x_j)g_{ij}$ is replaced by $(x_i - x_j - v_{ij})g_{ij}$ and the term $-v_{ij}g_{ij}$ is moved into the right-hand side of the system; similarly in j-th equation the term $(x_i - x_j)g_{ij}$ is replaced by $(x_j - x_i + v_{ij})g_{ij}$ and the term $v_{ij}g_{ij}$ is moved into the right-hand side of the system. As a result, we can calculate noise voltage U_{ij} due to resistor R_{ij} as

$$U_{ij} = (H_i - H_j)g_{ij}v_{ij}. \tag{58}$$

Corresponding spectral density $S_{ij}(f)$ is given by the formula:

$$\begin{aligned} S_{ij}(f) &= 4kTR_{ij}|H_i(2\pi f) - H_j(2\pi f)|^2 g_{ij}^2 \\ &= 4kT|g_{ij}| \cdot |H_i(2\pi f) - H_j(2\pi f)|^2. \end{aligned} \tag{59}$$

We can also rewrite (58) and (59) in matrix form, which gives

$$U_{ij} = \boldsymbol{H}_{cv}(\boldsymbol{e}_i - \boldsymbol{e}_j)\,g_{ij}v_{ij} \tag{60}$$

$$S_{ij}(f) = 4kT|g_{ii}| \cdot |\boldsymbol{H}_{cv}(2\pi f)(\boldsymbol{e}_i - \boldsymbol{e}_j)|^2. \tag{61}$$

Using (53), (57) and (61) one can easily calculate the total output noise voltage spectrum $S_R(f)$ of the filter in Fig. 1 due to its resistors

$$S_R(f) = \sum_{i=1}^{n}\left[S_{bi}(f) + \sum\nolimits_{j=1}^{i} S_{ij}(f)\right] \tag{62}$$

that is

$$\begin{aligned} S_R(f) = 4kT\sum_{i=1}^{n}\Bigg[&(|g_{bi}| + |g_{ii}|)\,|\boldsymbol{H}_{cv}(j2\pi f)\,\boldsymbol{e}_i|^2 \\ &+ \sum_{j=i+1}^{n}|g_{ij}| \cdot |\boldsymbol{H}_{cv}(2\pi f)(\boldsymbol{e}_i - \boldsymbol{e}_j)|^2\Bigg]. \end{aligned} \tag{63}$$

Finally, the total output noise voltage spectrum $S_{no}(f)$ of the filter in Fig. 1 can be calculated as a sum of $S_O(f)$ and $S_R(f)$:

$$\begin{aligned} S_{no}(f) &= S_O(f) + S_R(f) \\ &= \sum_{q=1}^{k} S_{Oq}(f)\,|H_{Oqo}(j2\pi f)|^2 g_{mq}^2 \\ &+ 4kT\sum_{i=1}^{n}\Bigg[(|g_{bi}| + |g_{ii}|)\,|\boldsymbol{H}_{cv}(j2\pi f)\,\boldsymbol{e}_i|^2 \\ &+ \sum_{j=i+1}^{n}|g_{ij}| \cdot |\boldsymbol{H}_{cv}(2\pi f)(\boldsymbol{e}_i - \boldsymbol{e}_j)|^2\Bigg]. \end{aligned} \tag{64}$$

Equivalent input referred noise voltage spectrum $S_{ni}(f)$ can be calculated by dividing $S_{no}(f)$ by $|H_{A,r_o}(j2\pi f)|^2$ – the square of modulus of filter's transfer function given by (8).

Note that in Eqs. (52)–(64) absolute values were taken where necessary to include the case when some of the matrix elements take negative values (see discussion after Eq. (9)). Note also that if formula (64) is to be applied to fully differential filter structure, noise spectrum of the filter resistors have to be counted twice, i.e. we have

$$\begin{aligned} S_{no}(f) &= S_O(f) + S_R(f) \\ &= \sum_{q=1}^{k} S_{Oq}(f)\,|H_{Oqo}(j2\pi f)|^2 g_{mq}^2 \\ &+ 8kT\sum_{i=1}^{n}\Bigg[(|g_{bi}| + |g_{ii}|)\,|\boldsymbol{H}_{cv}(j2\pi f)\,\boldsymbol{e}_i|^2 \\ &+ \sum_{j=i+1}^{n}|g_{ij}| \cdot |\boldsymbol{H}_{cv}(2\pi f)(\boldsymbol{e}_i - \boldsymbol{e}_j)|^2\Bigg]. \end{aligned} \tag{65}$$

Although formula (64) is quite complex, it can be easily evaluated numerically and can be used as a basis for automated noise analysis and optimization software.

In case of neglecting output resistance of filter amplifiers, we have to modify our equations. The vector $\boldsymbol{H}_{cv}$ now takes the form

$$\boldsymbol{H}_{cv}(s) = \boldsymbol{C}(\boldsymbol{P}_Y\boldsymbol{Y} + \boldsymbol{P}_G\boldsymbol{G})^{-1} \tag{66}$$

while the output noise U_{Oq} of the filter due to amplifier O_q is calculated as

$$U_{Oq} = H_{Oqo}A_q v_{Oq} \tag{67}$$

(recall that A_q is gain of O_q). Spectral density $S_{o.Oq}(f)$ is given by the formula:

$$S_{o.Oq}(f) = S_{Oq}(f)\,|H_{Oqo}(j2\pi f)|^2|A_q(f)|^2. \tag{68}$$

The total output noise voltage spectrum $S_O(f)$ of the filter in Fig. 1 due to its operational amplifiers can be now calculated as

$$S_O(f) = \sum_{q=1}^{k} S_{Oq}(f)\,|H_{Oqo}(j2\pi f)|^2|A_q(f)|^2. \tag{69}$$

In order to calculate the output noise of the filter in Fig. 1 due to resistors we use the same methodology as before.

In particular, the output noise voltage U_{bi} due to resistor R_{bi}, $i = 1, ..., n$, can be calculated from the equation

$$U_{bi} = \begin{cases} H_i g_{bi} v_{bi} & \text{if } O_{qo} \neq x_i, \quad q = 1, ..., k \\ 0 & \text{otherwise} \end{cases} \tag{70}$$

where v_{bi} is the noise voltage of R_{bi} (corresponding spectral density is $4kTR_{bi}$) and H_i is i-th component of the vector $\boldsymbol{H}_{cv}$ defined by (66). Note that U_{bi} is 0 whenever the output node of one of the filter OPAMPs is connected to the node x_i, since if we neglect output resistance of the OPAMP, then x_i is connected to the output of ideal voltage-controlled voltage source. Equation (70) can be rewritten in matrix form as follows:

$$U_{bi} = \boldsymbol{H}_{cv} \boldsymbol{P}_Y \boldsymbol{e}_i g_{bi} v_{bi} \tag{71}$$

where $\boldsymbol{e}_i$ is usual $n \times 1$ elementary vector. Corresponding spectral density $S_{bi}(f)$ is given by:

$$S_{bi}(f) = \begin{cases} 4kT|g_{bi}| \cdot |H_i(2\pi f)|^2 & \text{if } O_{qo} \neq x_i, \quad q = 1, ..., k \\ 0 & \text{otherwise} \end{cases} \tag{72}$$

or, in matrix form

$$S_{bi}(f) = 4kT|g_{bi}| \cdot |\boldsymbol{H}_{cv}(2\pi f) \boldsymbol{P}_Y \boldsymbol{e}_i|^2. \tag{73}$$

Output noise voltage U_{ii} due to grounded resistors R_{ii}, $i = 1, ..., n$, can be calculated as

$$U_{ii} = \begin{cases} H_i g_{ii} v_{ii} & \text{if } O_{qo} \neq x_i, \quad q = 1, ..., k \\ 0 & \text{otherwise} \end{cases} \tag{74}$$

where v_{ii} is the noise voltage of R_{ii} (corresponding spectral density is $4kTR_{ii}$) and H_i is i-th component of the vector $\boldsymbol{H}_{cv}$ defined by (66). Corresponding spectral density $S_{ii}(f)$ is given by the formula:

$$S_{ii}(f) = \begin{cases} 4kT|g_{ii}| \cdot |H_i(2\pi f)|^2 & \text{if } O_{qo} \neq x_i, \quad q = 1, ..., k \\ 0 & \text{otherwise} \end{cases}. \tag{75}$$

We can also rewrite (64) and (65) in matrix form, which gives

$$U_{ii} = \boldsymbol{H}_{cv} \boldsymbol{P}_Y \boldsymbol{e}_i g_{ii} v_{ii} \tag{76}$$

$$S_{ii}(f) = 4kT|g_{ii}| \cdot |\boldsymbol{H}_{cv}(2\pi f) \boldsymbol{P}_Y \boldsymbol{e}_i|^2. \tag{77}$$

Using similar reasoning as the one leading to (60), we can obtain the output noise voltage U_{ij} due to floating resistors R_{ij}, $i, j = 1, ..., n, \ i < j$:

$$U_{ij} = \begin{cases} (H_i - H_j) g_{ij} v_{ij} & \text{if } (O_{qo} \neq x_i, \quad q = 1, ..., k) \\ H_i g_{ij} v_{ij} & \text{if } (O_{qo} \neq x_i, \quad q = 1, ..., k) \\ & \wedge \ O_{qo} = x_j \ \text{ for some } \ q = 1, ..., k \\ -H_j g_{ij} v_{ij} & \text{if } (O_{qo} \neq x_j, \quad q = 1, ..., k) \\ & \wedge \ O_{qo} = x_i \ \text{ for some } \ q = 1, ..., k \\ 0 & \text{otherwise} \end{cases} \tag{78}$$

where v_{ij} is the noise voltage of R_{ij} (corresponding spectral density is $4kTR_{ij}$) while H_i and H_j are i-th and j-th components of the vector $\boldsymbol{H}_{cv}$ defined by (66). Corresponding spectral density $S_{ij}(f)$ is given by the formula:

$$S_{ij}(f) = \begin{cases} 4kT|g_{ij}| \cdot |H_i(2\pi f) - H_j(2\pi f)|^2 \\ \qquad \text{if } (O_{qo} \neq x_i, \quad q = 1, ..., k) \\ 4kT|g_{ij}| \cdot |H_i(2\pi f)|^2 \\ \qquad \text{if } (O_{qo} \neq x_i, \quad q = 1, ..., k) \\ \qquad \wedge \ O_{qo} = x_j \ \text{ for some } \ q = 1, ..., k \\ 4kT|g_{ij}| \cdot |H_j(2\pi f)|^2 \\ \qquad \text{if } (O_{qo} \neq x_j, \quad q = 1, ..., k) \\ \qquad \wedge \ O_{qo} = x_i \ \text{ for some } \ q = 1, ..., k \\ 0 \qquad \text{otherwise.} \end{cases} \tag{79}$$

We can also rewrite (78) and (79) in matrix form, which gives

$$U_{ij} = \boldsymbol{H}_{cv} \boldsymbol{P}_Y (\boldsymbol{e}_i - \boldsymbol{e}_j) g_{ij} v_{ij} \tag{80}$$

$$S_{ij}(f) = 4kT|g_{ii}| \cdot |\boldsymbol{H}_{cv}(2\pi f) \boldsymbol{P}_Y (\boldsymbol{e}_i - \boldsymbol{e}_j)|^2. \tag{81}$$

Using (73), (77) and (81) one can calculate the total output noise voltage spectrum $S_R(f)$ of the filter in Fig. 1 due to its resistors

$$S_R(f) = 4kT \sum_{i=1}^{n} \left[(|g_{bi}| + |g_{ii}|) |\boldsymbol{H}_{cv}(j2\pi f) \boldsymbol{P}_Y \boldsymbol{e}_i|^2 + \sum_{j=i+1}^{n} |g_{ij}| \cdot |\boldsymbol{H}_{cv}(2\pi f) \boldsymbol{P}_Y (\boldsymbol{e}_i - \boldsymbol{e}_j)|^2 \right]. \tag{82}$$

Finally, the total output noise voltage spectrum $S_{no}(f)$ of the filter in Fig. 1 can be calculated as a sum of $S_O(f)$ and $S_R(f)$:

$$\begin{aligned} S_{no}(f) &= S_O(f) + S_R(f) \\ &= \sum_{q=1}^{k} S_{Oq}(f) |H_{Oqo}(j2\pi f)|^2 |A_q(f)|^2 \\ &+ 4kT \sum_{i=1}^{n} \left[(|g_{bi}| + |g_{ii}|) |\boldsymbol{H}_{cv}(j2\pi f) \boldsymbol{P}_Y \boldsymbol{e}_i|^2 \right. \\ &\left. + \sum_{j=i+1}^{n} |g_{ij}| \cdot |\boldsymbol{H}_{cv}(2\pi f) \boldsymbol{P}_Y (\boldsymbol{e}_i - \boldsymbol{e}_j)|^2 \right]. \end{aligned} \tag{83}$$

Equivalent input referred noise voltage spectrum $S_{ni}(f)$ can be calculated by dividing $S_{no}(f)$ by $|H_A(j2\pi f)|^2$ – the square of modulus of filter's transfer function given by (14). Note also that if formula (83) is to be applied to fully differential filter structure, noise spectrum of the filter resistors have to be counted twice, i.e. we have to put $8kT$ instead of $4kT$.

Finally, if we assume infinite open-loop gain of filter amplifiers we have to consider usual identification of node voltages and reduction of equation number. Define the vector $\bar{\boldsymbol{H}}_{cv}$

$$\bar{\boldsymbol{H}}_{cv} = \left[\bar{H}_1 \cdots \bar{H}_{n-k} \right] = \bar{\boldsymbol{C}} \bar{\boldsymbol{Y}}^{-1} \tag{84}$$

where $\bar{\boldsymbol{Y}}$ and $\bar{\boldsymbol{C}}$ are given by (21) and (23), respectively. It can be shown that the output noise U_{oq} of the filter due to amplifier O_q can be calculated as follows

$$U_{Oq} = \bar{\boldsymbol{H}}_{cv} \bar{\boldsymbol{B}}_q v_{Oq} \tag{85}$$

where $\bar{\boldsymbol{B}}_q$ is the $\min\{O_{q+}, O_{q-}\}$-th column of matrix $\boldsymbol{LP}_Y\boldsymbol{Y}$ (if one of O_{q+}, O_{q-} is equal to x_g or x_0 then the other one is taken as a column number). Spectral density $S_{o.Oq}(f)$ is given by the formula:

$$S_{o.Oq}(f) = S_{Oq}(f)\,|\bar{\boldsymbol{H}}_{cv}(j2\pi f)\,\bar{\boldsymbol{B}}_q|^2. \tag{86}$$

The total output noise voltage spectrum $S_O(f)$ of the filter due to operational amplifiers can be now calculated as

$$S_O(f) = \sum_{q=1}^{k} S_{Oq}(f)\,|\bar{\boldsymbol{H}}_{cv}(j2\pi f)\,\bar{\boldsymbol{B}}_q|^2. \tag{87}$$

Now we calculate the output noise of the filter in Fig. 1 due to resistors. In particular, the output noise voltage U_{bi} due to resistor R_{bi}, $i = 1, ..., n$, can be calculated from the equation

$$U_{bi} = \bar{\boldsymbol{H}}_{cv}\boldsymbol{LP}_Y\boldsymbol{e}_i g_{bi} v_{bi} \tag{88}$$

where v_{bi} is the noise voltage of R_{bi} (corresponding spectral density is $4kTR_{bi}$). Note that U_{bi} is 0 whenever the output node of one of the filter OPAMPs is connected to the node x_i, since if we neglect output resistance of the OPAMP, then x_i is connected to the output of ideal voltage-controlled voltage source. Corresponding spectral density $S_{bi}(f)$ is given by:

$$S_{bi}(f) = 4kT|g_{bi}|\cdot|\bar{\boldsymbol{H}}_{cv}(2\pi f)\,\boldsymbol{LP}_Y\boldsymbol{e}_i|^2. \tag{89}$$

Output noise voltage U_{ii} due to grounded resistors R_{ii}, $i = 1, ..., n$, can be calculated as

$$U_{ii} = \bar{\boldsymbol{H}}_{cv}\boldsymbol{LP}_Y\boldsymbol{e}_i g_{ii} v_{ii} \tag{90}$$

where v_{ii} is the noise voltage of R_{ii} (corresponding spectral density is $4kTR_{ii}$). Corresponding spectral density $S_{ii}(f)$ is given by the formula:

$$S_{ii}(f) = 4kT|g_{ii}|\cdot|\bar{\boldsymbol{H}}_{cv}(2\pi f)\,\boldsymbol{LP}_Y\boldsymbol{e}_i|^2. \tag{91}$$

The output noise voltage U_{ij} due to floating resistors R_{ij}, $i, j = 1, ..., n$, $i < j$ is given by

$$U_{ij} = \bar{\boldsymbol{H}}_{cv}\boldsymbol{LP}_Y(\boldsymbol{e}_i - \boldsymbol{e}_j)\,g_{ij} v_{ij} \tag{92}$$

where v_{ij} is the noise voltage of R_{ij} (corresponding spectral density is $4kTR_{ij}$). Corresponding spectral density $S_{ij}(f)$ is given by the formula:

$$S_{ij}(f) = 4kT|g_{ii}|\cdot|\bar{\boldsymbol{H}}_{cv}(2\pi f)\,\boldsymbol{LP}_Y(\boldsymbol{e}_i - \boldsymbol{e}_j)|^2. \tag{93}$$

Using (89), (91) and (93) one can calculate the total output noise voltage spectrum $S_R(f)$ of the filter in Fig. 1 due to its resistors

$$S_R(f) = 4kT\sum_{i=1}^{n}\Bigg[(|g_{bi}| + |g_{ii}|)\,|\bar{\boldsymbol{H}}_{cv}(j2\pi f)\,\boldsymbol{LP}_Y\boldsymbol{e}_i|^2 + \sum_{j=i+1}^{n}|g_{ij}|\cdot|\bar{\boldsymbol{H}}_{cv}(2\pi f)\,\boldsymbol{LP}_Y(\boldsymbol{e}_i - \boldsymbol{e}_j)|^2\Bigg]. \tag{94}$$

Finally, the total output noise voltage spectrum $S_{no}(f)$ of the filter in Fig. 1 can be calculated as a sum of $S_O(f)$ and $S_R(f)$:

$$S_{no}(f) = S_O(f) + S_R(f) = \sum_{q=1}^{k} S_{Oq}(f)\,|\bar{\boldsymbol{H}}_{cv}(j2\pi f)\,\bar{\boldsymbol{B}}_q|^2 + 4kT\sum_{i=1}^{n}\Bigg[(|g_{bi}| + |g_{ii}|)\,|\bar{\boldsymbol{H}}_{cv}(j2\pi f)\,\boldsymbol{LP}_Y\boldsymbol{e}_i|^2 + \sum_{j=i+1}^{n}|g_{ij}|\cdot|\bar{\boldsymbol{H}}_{cv}(2\pi f)\,\boldsymbol{LP}_Y(\boldsymbol{e}_i - \boldsymbol{e}_j)|^2\Bigg]. \tag{95}$$

Equivalent input referred noise voltage spectrum $S_{ni}(f)$ can be calculated by dividing $S_{no}(f)$ by $|H(j2\pi f)|^2$ – the square of modulus of filter's transfer function given by (22). Note also that if formula (95) is to be applied to fully differential filter structure, noise spectrum of the filter resistors have to be counted twice, i.e. we have to put $8kT$ instead of $4kT$.

For the sake of illustration consider evaluation of noise formula (95) for the second-order filter in Fig. 5. We omit formulas (65) and (83) which allow to evaluate filter noise while taking into consideration finite OPAMP gain and non-zero output resistance because they lead to really long formulas even in this simple case. In practice, of course, these formulas are evaluated by appropriate computer software.

In order to evaluate formula (95) we need to know vector $\bar{\boldsymbol{H}}_{cv} = \bar{\boldsymbol{C}}\bar{\boldsymbol{Y}}^{-1}$ as well as vectors $\bar{\boldsymbol{B}}_q$, $q = 1, 2$. Using (21), (23)–(25) and (84) we obtain:

$$\bar{\boldsymbol{H}}_{cv} = \frac{1}{D(s)}\begin{bmatrix} R_2^{-1} & -sC_1 \end{bmatrix} \tag{96}$$

$$\bar{\boldsymbol{B}}_1 = \begin{bmatrix} R_b^{-1} + R_1^{-1} + sC_1 \\ 0 \end{bmatrix}, \quad \bar{\boldsymbol{B}}_2 = \begin{bmatrix} 0 \\ R_2^{-1} + R_3^{-1} + sC_2 \end{bmatrix} \tag{97}$$

where $D(s) = C_1C_2s^2 + C_1R_3^{-1}s + R_1^{-1}R_2^{-1}$. We also assume that input referred noise spectrum of both filter OPAMPs is the same, independent of frequency, and equal to S_n. The output noise spectrum of the filter in Fig. 5 can be then calculated as

$$\begin{aligned} S_{no}(f) = {} & \frac{S_n}{|D(j2\pi f)|^2}\big[|R_b^{-1} + R_1^{-1} + j2\pi fC_1|^2 \\ & \times R_2^{-2} + |R_2^{-1} + R_3^{-1} + j2\pi fC_2|^2|j2\pi fC_1|^2\big] \\ & + \frac{8kT}{|D(j2\pi f)|^2}\big[\left(R_b^{-1} + R_1^{-1}\right)R_2^{-2} \\ & + \left(R_2^{-1} + R_3^{-1}\right)|j2\pi fC_1|^2\big] \end{aligned} \tag{98}$$

and the corresponding input referred noise spectrum

$$\begin{aligned} S_{ni}(f) = {} & S_nR_b^2R_2^2\big[|R_b^{-1} + R_1^{-1} + j2\pi fC_1|^2 \\ & \times R_2^{-2} + |R_2^{-1} + R_3^{-1} + j2\pi fC_2|^2|j2\pi fC_1|^2\big] \\ & + 8kTR_b^2R_2^2\big[\left(R_b^{-1} + R_1^{-1}\right)R_2^{-2} \\ & + \left(R_2^{-1} + R_3^{-1}\right)|j2\pi fC_1|^2\big]. \end{aligned} \tag{99}$$

If we assume that all filter resistors are the same and equal R, the above formulas take the form:

$$S_{no}(f) = \frac{S_n}{|D\,(j2\pi f)\,|^2}\left[|2R^{-1}+j2\pi fC_1|^2 \right.$$
$$\left. \times R^{-2} + |2R^{-1}+j2\pi fC_2|^2|j2\pi fC_1|^2\right] \qquad (100)$$
$$+\frac{8kT}{|D\,(j2\pi f)\,|^2}\left[2R^{-3}+2R^{-1}|j2\pi fC_1|^2\right]$$

$$S_{ni}(f) = S_n\big\lfloor|2+j2\pi fC_1R|^2$$
$$+|2+j2\pi fC_2R|^2|j2\pi fC_1R|^2\big\rfloor \qquad (101)$$
$$+8kTR\big\lfloor 2+R^2|j2\pi fC_1|^2\big\rfloor.$$

5. Verification

For the sake of verification we have compared theoretical results with SPICE simulation using two example filters: a 3rd order low-pass Chebyshev filter in leap-frog structure with 3 dB frequency equal to 12 MHz (specifications typical for VDSL filters [5]) shown in Fig. 9, and a 5th order low-pass Butterworth filter in leap-frog structure with 3 dB frequency equal to 5 MHz shown in Fig. 10. The filters are realized using simple two-stage class AB OPAMP with Miller compensation shown in Fig. 11. The circuit is implemented in standard 0.35 μm CMOS process. Simulated OPAMP parameters are: DC gain – 71 dB, open-loop 3 dB frequency – 430 kHz, phase margin 45°, input referred noise spectrum 14.5 nV/Hz$^{1/2}$ (only thermal noise is considered), output resistance $ro = 15$ kΩ. Filter elements are $C_1 = 2.28$ pF, $C_2 = 2.01$ pF, $C_3 = 1.53$ pF, $R_i = R = 10$ kΩ, $i = 1, ..., 6$ (Chebyshev filter), and $C_1 = 5.10$ pF, $C_2 = 5.59$ pF, $C_3 = 4.56$ pF, $C_4 = 2.95$ pF, $C_5 = 1.02$ pF, $R_i = R = 10$ kΩ, $i = 1, ..., 10$ (Butterworth filter). Figures 12 and 13 show theoretical and simulated frequency responses of the filters in Figs. 9 and 10, respectively. We can observe nominal (ideal) response as well as actual response that is distorted due to the finite gain-bandwidth product (GBW) and non-zero output resistance of OPAMPs. The agreement between theoretical and simulated data is very good. Figures 14–17 show theoretical and simulated input/output referred noise spectrum of the filters in Figs. 9 and 10, respectively. Also in this case, the agreement between both sets of data is excellent.

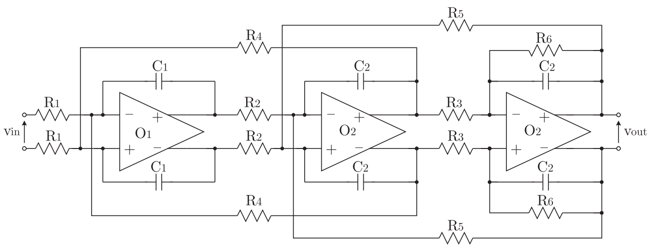

Fig. 9. Fully differential 3rd order leap-frog Active-RC filter

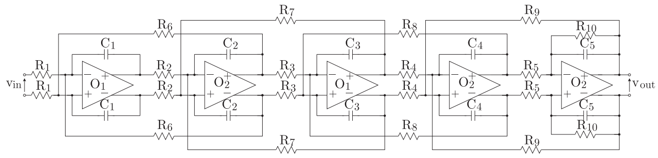

Fig. 10. Fully differential 5th order leap-frog Active-RC filter

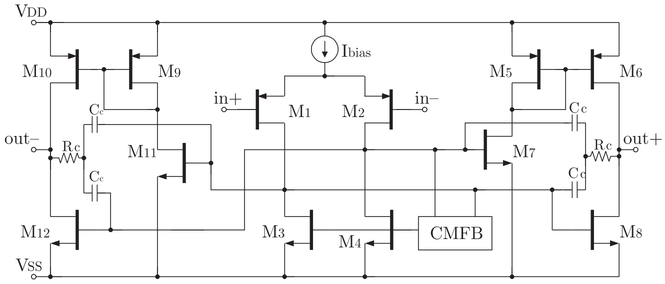

Fig. 11. Simple class AB fully-differential OPAMP

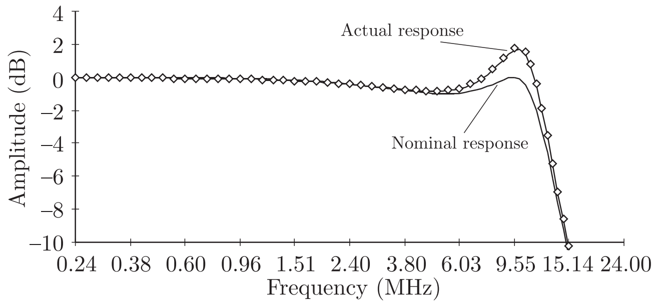

Fig. 12. Frequency response of 3rd order 1dB Chebyshev filter in Fig. 9: nominal and actual response; theory (solid line), and simulation (points)

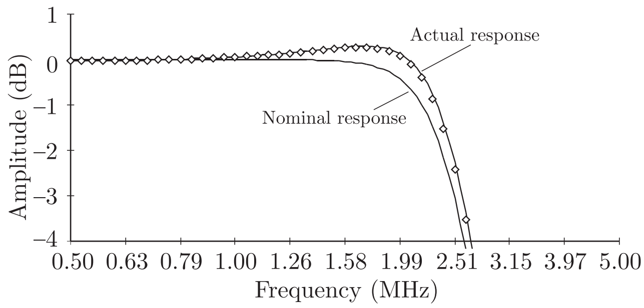

Fig. 13. Frequency response of 5th order Butterworth filter in Fig. 10: nominal and actual response; theory (solid line), and simulation (points)

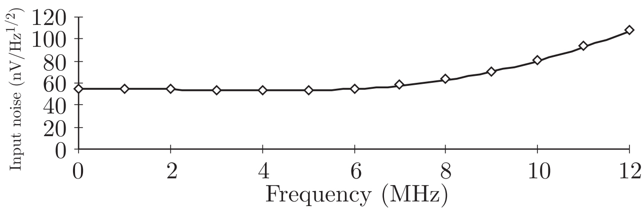

Fig. 14. Input referred noise spectrum of the filter in Fig. 9; theory (solid line), and simulation (points)

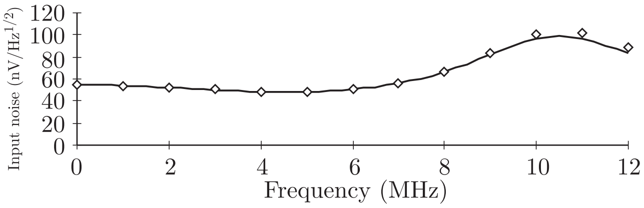

Fig. 15. Output referred noise spectrum of the filter in Fig. 9; theory (solid line), and simulation (points)

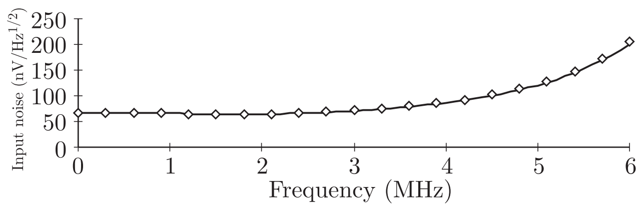

Fig. 16. Input referred noise spectrum of the filter in Fig. 10; theory (solid line), and simulation (points)

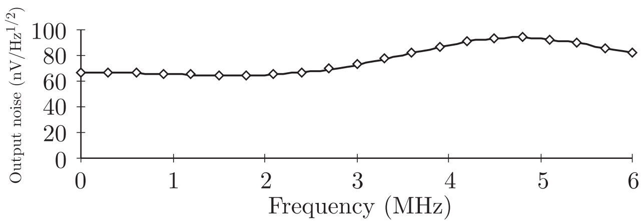

Fig. 17. Output referred noise spectrum of the filter in Fig. 10; theory (solid line), and simulation (points)

6. Conclusions

A general topology of continuous-time Active-RC filter that includes all possible structures of filters of this class (both single-ended and fully-differential) is presented. The model is analyzed using a unified algebraic formalism, which makes it suitable for use in computer-aided analysis and design of Active-RC filters. The model takes into account finite DC gain and finite bandwidth as well as non-zero output resistance of operational amplifiers. It is further used to evaluate filter noise, and sensitivity. The accuracy of the model is verified by comparing theoretical results to SPICE simulations. The goal of the future work is to develop - within the presented approach - tools for investigating nonlinear effects in Active-RC filters as well as to use the model for computer-aided design and optimization of filters of this class.

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