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Structure decomposition of normal 2D transfer matrices

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Abstract. The notion of the normal transfer matrix and the notion of the structure decomposition of normal transfer matrix for 2D general model are introduced. Necessary and sufficient conditions for the existence of the structure decomposition of normal transfer matrix are established. A procedure for computation of the structure decomposition is proposed and illustrated by the numerical example. It is shown that the impulse response matrix of the normal model is independent of the polynomial part of its structure decomposition.

Keywords: normal transfer matrix, structure decomposition, 2D system, solvability condition, procedure.

1. Introduction

Lampe and Rosenwasser in [1,2] have introduced the notions of the normal matrix and the structure decomposition (S-Darstellung) of normal matrices. They have shown that if the normal matrix is written in the standard form $T = N/d$ (d is the minimal common denominator) then every second order nonzero minor of the polynomial matrix N is divisible (with zero remainder) by the polynomial d. They have also shown that there exists a structure decomposition of transfer matrices if and only if the matrices are normal. The influence of the state-feedback on cyclity of linear systems and the normalization of linear systems by state-feedbacks is considered in [3,4]. Some implications of the notion of the normal matrix on electrical circuit is discussed in [5].

The concept of the normal matrix and the structure decomposition of normal matrices have been extended for standard positive systems in [6].

The most popular models of two-dimensional (2D) linear systems are the models introduced by Roesser [7,8,9], Fornasini-Marchesini [10,11] and Kurek [12].

In this paper the notion of normal matrix and its structure decomposition will be extended for 2D general model. Necessary and sufficient conditions for the existence of the structure decomposition of the normal 2D general model are be established and a procedure for computation of the structure decomposition is given. It is shown that the impulse response matrix of the normal model is independent of the polynomial part of its structure decomposition.

2. Preliminaries and normal matrices

Let $R^{m\times n}$ be the set of $m \times n$ real matrices and $R^m = R^{m\times 1}$. The set of nonnegative integers is denoted by Z_+.

Consider the general 2D model [12]

$$x_{i+1,j+1} = A_0x_{ij} + A_1x_{i+1,j} + A_2x_{i,j+1} + B_0u_{ij} + B_1u_{i+1,j} + B_2u_{i,j+1} \tag{1a}$$

$$y_{ij} = Cx_{ij} + Du_{ij} \tag{1b}$$

where $x_{ij} \in R^n$, $u_{ij} \in R^m$, $y_{ij} \in R^p$ are the state, input and output vectors and $A_k \in R^{n\times n}$, $B_k \in R^{n\times m}$, $k = 0, 1, 2$, $C \in R^{p\times n}$, $D \in R^{p\times m}$.

The boundary conditions for (1a) are given by

$$x_{i0} \text{ for } i \in Z_+ \text{ and } x_{0j} \text{ for } j \in Z_+. \tag{2}$$

The transfer matrix of (1) has the form

$$T(z_1, z_2) = C[Iz_1z_2 - A_0 - A_1z_1 - A_2z_2]^{-1}(B_0 + B_1z_1 + B_2z_2) + D. \tag{3}$$

The matrix (3) can be written in the form

$$T(z_1, z_2) = \frac{N(z_1, z_2)}{d(z_1, z_2)} \tag{4}$$

where $N(z_1, z_2) \in R^{p\times m}[z_1, z_2]$ (the set of $p \times m$ polynomial matrices) and $d(z_1, z_2)$ is the minimal common denominator.

The matrix $T(z_1, z_2)$ is irreducible if and only if for any zero (z_1^0, z_2^0) of $d(z_1, z_2)$ $(d(z_1^0, z_2^0) = 0)$ we have $N(z_1^0, z_2^0) \neq 0$.

The matrix (4) has the standard form if and only if it is irreducible and the polynomial

$$d(z_1, z_2) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} d_{ij}z_1^iz_2^j \tag{5}$$

is a monic polynomial, i.e. $d_{n_1n_2} = 1$.

Definition 1. The standard matrix (4) with $\min(p, m) \geqslant 2$ is called normal if and only if every nonzero second order minor of the polynomial matrix $N(z_1, z_2)$ is divisible (with zero remainder) by the polynomial $d(z_1, z_2)$.

3. Structure decomposition

Let us assume that the polynomial matrix $N(z_1, z_2)$ of (4) can be written in the form

$$N(z_1, z_2) = P(z_1, z_2)Q(z_1, z_2) + d(z_1, z_2)G(z_1, z_2) \tag{6}$$

where $P(z_1, z_2) \in R^p[z_1, z_2]$, $Q(z_1, z_2) \in R^{1\times m}[z_1, z_2]$, $G(z_1, z_2) \in R^{p\times m}[z_1, z_2]$ and $d(z_1, z_2)$ is the minimal common denominator of (4).

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Substitution of (6) into (4) yields

$$T(z_1,z_2)=\frac{P(z_1,z_2)Q(z_1,z_2)}{d(z_1,z_2)}+G(z_1,z_2). \tag{7}$$

DEFINITION 2. It is said that there exists a structure decomposition of $T(z_1,z_2)$ if and only if it can be written in the form (7).

In what follows the following row and column operations [7] will be used:

1. Multiplication of any row (column) by a nonzero real number
2. Addition to any row (column) of any other row (column) multiplied by any polynomial $p(z_1,z_2)$
3. Interchange of any two rows (columns).

Let us assume that applying the elementary row and column operations gives the possibility to reduce the polynomial matrix $N(z_1,z_2)$ to the form

$$U(z_1,z_2)N(z_1,z_2)V(z_1,z_2)=i(z_1,z_2)\begin{bmatrix}1 & r(z_1,z_2)\\ c(z_1,z_2) & \bar{N}(z_1,z_2)\end{bmatrix} \tag{8}$$

where $U(z_1,z_2)\in R^{p\times p}[z_1,z_2]$, $V(z_1,z_2)\in R^{m\times m}[z_1,z_2]$ are unimodular matrices ($\det U(z_1,z_2)=\alpha\neq 0$, $\det V(z_1,z_2)=\beta\neq 0$, α,β-real numbers) of the elementary row and column operations, respectively and $i(z_1,z_2)\in R[z_1,z_2]$, $r(z_1,z_2)\in R^{1\times(m-1)}[z_1,z_2]$, $c(z_1,z_2)\in R^{p-1}[z_1,z_2]$, $\bar{N}(z_1,z_2)\in R^{(p-1)\times(m-1)}[z_1,z_2]$.

THEOREM 1. Let the polynomial matrix $N(z_1,z_2)$ can be reduced to the form (8). Then there exists a structure decomposition (7) of $T(z_1,z_2)$ if and only if the matrix (4) is normal.

P r o o f . *Necessity.* Let $N_{k,l}^{i,j}(z_1,z_2)$ be the second order minor composed of the i-th and j-th rows and k-th and l-th columns of the matrix $N(z_1,z_2)$. If there exists a structure decomposition (7) then (6) holds and (Eq. 9) where $p_i(z_1,z_2)$, $q_k(z_1,z_2)$ and $g_{ik}(z_1,z_2)$ are the entries of the matrices $P(z_1,z_2)$, $G(z_1,z_2)$ and $n_{k,l}^{i,j}(z_1,z_2)$, respectively and is a polynomial.

From (9), shown at the bottom of the page, it follows that the minor $N_{k,l}^{i,j}(z_1,z_2)$ is divisible by $d(z_1,z_2)$. Therefore, by definition 1 the matrix (4) is normal.

Sufficiency. If the matrix (4) is normal and (8) holds then every nonzero second order minor of the matrix (8) is divisible by $d(z_1,z_2)$ since by the Benet-Cauchy theorem [2] every second order minor of $U(z_1,z_2)N(z_1,z_2)V(z_1,z_2)$ is the sum of products of second order minors of the matrices $U(z_1,z_2)$, $N(z_1,z_2)$ and $V(z_1,z_2)$. Hence we have

$$i(z_1,z_2)\left[\bar{N}(z_1,z_2)-c(z_1,z_2)r(z_1,z_2)\right]=d(z_1,z_2)\hat{N}(z_1,z_2) \tag{10}$$

for some $\hat{N}(z_1,z_2)\in R^{(p-1)\times(m-1)}[z_1,z_2]$.

Defining

$$\begin{aligned}P(z_1,z_2)&=U^{-1}(z_1,z_2)i(z_1,z_2)\begin{bmatrix}1\\ c(z_1,z_2)\end{bmatrix}\\ Q(z_1,z_2)&=\begin{bmatrix}1 & r(z_1,z_2)\end{bmatrix}V^{-1}(z_1,z_2)\\ G(z_1,z_2)&=U^{-1}(z_1,z_2)\begin{bmatrix}0 & 0\\ 0 & d(z_1,z_2)\hat{N}(z_1,z_2)\end{bmatrix}\end{aligned} \tag{11}$$

from (8), (10) and (11) we obtain

$$\begin{aligned}&N(z_1,z_2)\\ &=U^{-1}(z_1,z_2)i(z_1,z_2)\begin{bmatrix}1 & r(z_1,z_2)\\ c(z_1,z_2) & \bar{N}(z_1,z_2)\end{bmatrix}V^{-1}(z_1,z_2)=\\ &=U^{-1}(z_1,z_2)\left\{i(z_1,z_2)\begin{bmatrix}1\\ c(z_1,z_2)\end{bmatrix}\begin{bmatrix}1 & r(z_1,z_2)\end{bmatrix}\right.\\ &\quad\left.+\begin{bmatrix}0 & 0\\ 0 & d(z_1,z_2)\hat{N}(z_1,z_2)\end{bmatrix}\right\}V^{-1}(z_1,z_2)=\\ &=P(z_1,z_2)Q(z_1,z_2)+d(z_1,z_2)G(z_1,z_2).\end{aligned}$$

Therefore, there exists the composition (7) of the matrix $T(z_1,z_2)$. □

THEOREM 2. If there exists a structure decomposition of the inverse matrix

$$[Iz_1z_2-A_0-A_1z_1-A_2z_2]^{-1}=\frac{\bar{P}(z_1,z_2)\bar{Q}(z_1,z_2)}{d(z_1,z_2)}+\bar{G}(z_1,z_2) \tag{12}$$

then there exists also the structure decomposition (7) of the transfer matrix (3) and the matrices of structure decompositions are related by

$$\begin{aligned}P(z_1,z_2)&=C\bar{P}(z_1,z_2),\; Q(z_1,z_2)=\bar{Q}(z_1,z_2)(B_0+B_1z_1+B_2z_2)\\ G(z_1,z_2)&=C\bar{G}(z_1,z_2)(B_0+B_1z_1+B_2z_2)+D.\end{aligned} \tag{13}$$

P r o o f . Substitution of (12) into (3) yields

$$\begin{aligned}&T(z_1,z_2)\\ &=C\left[\frac{\bar{P}(z_1,z_2)\bar{Q}(z_1,z_2)}{d(z_1,z_2)}+\bar{G}(z_1,z_2)\right](B_0+B_1z_1+B_2z_2)\\ &\quad+D=\\ &=\frac{P(z_1,z_2)Q(z_1,z_2)}{d(z_1,z_2)}+G(z_1,z_2)\end{aligned}$$

where $P(z_1,z_2)$, $Q(z_1,z_2)$ and $G(z_1,z_2)$ are defined by (13).

$$\begin{aligned}N_{k,l}^{i,j}(z_1,z_2)&=\begin{vmatrix}p_i(z_1,z_2)q_k(z_1,z_2)+d(z_1,z_2)g_{ik}(z_1,z_2) & p_i(z_1,z_2)q_l(z_1,z_2)+d(z_1,z_2)g_{il}(z_1,z_2)\\ p_j(z_1,z_2)q_k(z_1,z_2)+d(z_1,z_2)g_{jk}(z_1,z_2) & p_j(z_1,z_2)q_l(z_1,z_2)+d(z_1,z_2)g_{jl}(z_1,z_2)\end{vmatrix}=\\ &=d(z_1,z_2)n_{kl}^{ij}(z_1,z_2)\quad\text{for } i,j=1,\dots,p,\; k,l=1,\dots,m\end{aligned} \tag{9}$$

If the assumptions of theorem 1 are satisfied then the structure decomposition (7) of $T(z_1, z_2)$ can be found by the use of the following procedure.

PROCEDURE

Step 1 Using the elementary row and column operations reduce the polynomial matrix $N(z_1, z_2)$ to the form (8) and find the unimodular matrices $U(z_1, z_2)$, $V(z_1, z_2)$ and $i(z_1, z_2)$, $r(z_1, z_2)$, $c(z_1, z_2)$ and $\bar{N}(z_1, z_2)$.

Step 2 Using (10) find the polynomial matrix $\hat{N}(z_1, z_2)$.

Step 3 Using (11) find the matrices $P(z_1, z_2)$, $Q(z_1, z_2)$ and $G(z_1, z_2)$.

Step 4 Using (7) find the desired structure decomposition.

Remark 1. The structure decomposition (7) of $T(z_1, z_2)$ can be also found by the following two steps procedure:

Step 1 Find the structure decomposition (12) of the matrix $[Iz_1z_2 - A_0 - A_1z_1 - A_2z_2]^{-1}$

Step 2 Using (13) find the desired decomposition (7)

Remark 2. From (6) and (11) it follows that if $i(z_1, z_2) = 1$ and there exists a structure decomposition (7) of (4) then

$$\operatorname{rank} N(z_1^0, z_2^0) = 1 \tag{14}$$

for every pair (z_1^0, z_2^0) satifynig $d(z_1^0, z_2^0) = 0$.

The condition (14) is the necessary condition for the existence of the structure decomposition (7) of (4).

Example 1. Consider the model (1) with

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B_1 = B_2 = 0,$$
$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ D = 0 \tag{15}$$

In this case the transfer matrix (3) has the form Eqs. 16, 17, 18, shown at the bottom of the page.

The matrix (16) is normal since the nonzero second order minors

$$M_1 = \begin{vmatrix} (z_1z_2 - z_2 - 1)(z_1z_2 - 1) & -z_2(z_1z_2 - 1) \\ 0 & z_1(z_2 - 1)(z_1z_2 - 1) \end{vmatrix},$$
$$M_2 = \begin{vmatrix} (z_1z_2 - z_2 - 1)(z_1z_2 - 1) & z_1z_2 \\ 0 & -z_1^2(z_2 - 1) \end{vmatrix}$$

are divisible by the polynomial (18).

The condition (14) is satisfied since for $z_1 = 0$, $z_2 = 1$ and $z_1 = z_2 = 1$ the matrix (17) has rank equal to 1.

Using the procedure we obtain.

Step 1. In this case

$$U(z_1, z_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$V(z_1, z_2) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 - z_1z_2 & 0 & 1 \end{bmatrix} \tag{19}$$

since 19a, shown at the bottom of the page.

Hence

$$\begin{aligned} &i(z_1, z_2) = 1, r(z_1, z_2) = [\, z_2\,(1 - z_1z_2) \quad z_1z_2 \,], \\ &c(z_1, z_2) = z_1\,(z_2 - 1)\left(z_1^2z_2 - z_1z_2 - 2z_1 + 1\right) \\ &\bar{N}(z_1, z_2) = [\, z_1\,(z_2 - 1)\,(z_1z_2 - 1) \quad z_1^2\,(1 - z_2) \,]. \end{aligned}$$

Step 2. Using (10) and (19) we obtain

$$\bar{N}(z_1, z_2) - c(z_1, z_2)r(z_1, z_2) = \hat{N}(z_1, z_2)d(z_1, z_2)$$

where

$$\hat{N}(z_1, z_2) = [\, 1 - z_1z_2 \quad -z_1 \,]. \tag{20}$$

Step 3. Using (11), (18), (19) and (20) we obtain (20a), shown at the top of the next page.

Step 4. The desired structure decomposition of (16) has the form (21), shown at the top of the next page.

$$\begin{aligned} T(z_1, z_2) &= C[Iz_1z_2 - A_0 - A_1z_1 - A_2z_2]^{-1}B_0 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1z_2 - z_1 & z_2 & 0 \\ 0 & z_1z_2 - z_2 - 1 & z_1 \\ 0 & 0 & z_1z_2 - 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{N(z_1, z_2)}{d(z_1, z_2)} \end{aligned} \tag{16}$$

where

$$N(z_1, z_2) = \begin{bmatrix} (z_1z_2 - z_2 - 1)(z_1z_2 - 1) & -z_2(z_1z_2 - 1) & z_1z_2 \\ 0 & z_1(z_2 - 1)(z_1z_2 - 1) & -z_1^2(z_2 - 1) \end{bmatrix} \tag{17}$$

$$d(z_1, z_2) = z_1(z_2 - 1)(z_1z_2 - z_2 - 1)(z_1z_2 - 1). \tag{18}$$

$$U(z_1, z_2)N(z_1, z_2)V(z_1, z_2) = \begin{bmatrix} 1 & z_2\,(1 - z_1z_2) & z_1z_2 \\ z_1\,(z_2 - 1)\left(z_1^2z_2 - z_1z_2 - 2z_1 + 1\right) & z_1\,(z_2 - 1)\,(z_1z_2 - 1) & z_1^2\,(1 - z_2) \end{bmatrix}. \tag{19a}$$

$$
\begin{aligned}
P(z_1,z_2) &= U^{-1}(z_1,z_2)i(z_1,z_2)\begin{bmatrix}1\\ c(z_1,z_2)\end{bmatrix} = \begin{bmatrix}1\\ z_1(z_2-1)\left(z_1^2z_2-z_1z_2-2z_1+1\right)\end{bmatrix}\\
Q(z_1,z_2) &= [\,1 \quad r(z_1,z_2)\,]\,V^{-1}(z_1,z_2) = [\,z_1^2z_2^2-z_1z_2^2-2z_1z_2+z_2-1 \quad -z_1z_2^2+z_2 \quad z_1z_2\,]\\
G(z_1,z_2) &= U^{-1}(z_1,z_2)\begin{bmatrix}0 & 0\\ 0 & d(z_1,z_2)\hat{N}(z_1,z_2)\end{bmatrix} =\\
&= \begin{bmatrix}0 & 0 & 0\\ 0 & z_1(1-z_2)(z_1z_2-z_2-1)(1-z_1z_2)^2 & z_1^2(1-z_2)(z_1z_2-z_2-1)(z_1z_2-1)\end{bmatrix}.
\end{aligned} \tag{20a}
$$

$$
\begin{aligned}
T(z_1,z_2) = {} & \frac{1}{z_1(z_2-1)(z_1z_2-z_2-1)(z_1z_2-1)}\begin{bmatrix}1\\ z_1(z_2-1)\left(z_1^2z_2-z_1z_2-2z_1+1\right)\end{bmatrix}\\
& [\,z_1^2z_2^2-z_1z_2^2-2z_1z_2+z_2-1 \quad -z_1z_2^2+z_2 \quad z_1z_2\,]\\
& +\begin{bmatrix}0 & 0 & 0\\ 0 & z_1(1-z_2)(z_1z_2-z_2-1)(1-z_1z_2)^2 & z_1^2(1-z_2)(z_1z_2-z_2-1)(z_1z_2-1)\end{bmatrix}.
\end{aligned} \tag{21}
$$

4. Impulse response matrix

It is well known [7] that the impulse response matrix g_{ij} is the orginal of the transfer matrix $T(z_1,z_2)$, i.e.

$$g_{ij} = Z^{-1}\left[T(z_1,z_2)\right] \tag{22}$$

where Z^{-1} is the 2D Z-transform inverse operator.

Let us consider the general 2D model (1) with $D=0$. The model (1) is called normal if there exists the structure decomposition (7) of its transfer matrix.

THEOREM 3. The impulse response matrix g_{ij} of the normal model (1) with $D=0$ is independent of the polynomial matrix $G(z_1,z_2)$ of the structure decomposition (7).

P r o o f. From (22) and the definition of 2D Z-transform we have

$$T(z_1,z_2) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} g_{ij} z_1^{-i} z_2^{-j} \tag{23}$$

If $D=0$ then the transfer matrix $T(z_1,z_2)$ of the normal model (1) is strictly proper. Note that the expansion of the first term of (7), i.e.

$$\frac{P(z_1,z_2)Q(z_1,z_2)}{d(z_1,z_2)} \tag{24}$$

can only give a series with nonpositive powers in z_1 and z_2. Therefore, the impulse response matrix is independent of $G(z_1,z_2)$.

Example 2. (Continuation of example 1). The transfer matrix of the model (1) with (15) has the form (16) and it is strictly proper. It is easy to check that the impulse response matrix of the model is independent of the polynomial matrix (24a).

$$G(z_1,z_2) = \begin{bmatrix}0 & 0 & 0\\ 0 & z_1(1-z_2)(z_1z_2-z_2-1)(1-z_1z_2)^2 & z_1^2(1-z_2)(z_1z_2-z_2-1)(z_1z_2-1)\end{bmatrix}. \tag{24a}$$

5. Concluding remarks

The notion of the normal transfer matrix and the notion of the structure decomposition of the normal transfer matrix for 2D general model have been introduced. Necessary and sufficient conditions for the existence of the structure decomposition of normal transfer matrix have been established. A procedure for computation of the structure decomposition has been presented and illustrated by numerical example. It has been shown that the impulse response matrix of the normal general model (1) is independent of the polynomial matrix $G(z_1,z_2)$ of its structure decomposition (7). The structure decomposition can be used for computation of minimal realization of given transfer matrices of the 2D general model is a similar way as it has been proposed for 1D case in [6].

An extension of these considerations for 2D positive general model is an open problem.

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