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Invited paper

Swing-up time analysis of pendulum

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Abstract. Swing-up control of a single pendulum from the pendant to the upright position is firstly surveyed. The control laws are comparatively studied based on swing-up time from a given initial state to the upright position. The State Dependent Riccati Equation is found effective for designing the swing-up control law under saturating control input. The control law is extended to a linear combination of sine function of the angle and the angular velocity, and a variable structure control with a sliding mode given by the linear combination. Making the swing-up time correspond to a colour, which is similar to the Fractal analysis, colour maps of the swing-up time for given control parameters and initial conditions yield interesting Fractal-like figures.

Keywords: pendulum control, state dependent Riccati equation, nonlinear control, artificial gravity, fractal.

1. Introduction

The equiperiodic swing motion of a hanging lamp (pendulum) is said to be first observed by Galilei at his age of 19 in the cathedral in Pisa, which led him to design a pendulum clock. Foucault used a pendulum to demonstrate the rotation of the earth. A hanging pendulum thus has interesting and attractive characteristics, and has been studied.

Stabilizing a pendulum at the upright unstable position has become an interesting object for physicists. Such stabilization was first done using feedforward vertical vibration of the pivot of a pendulum by A. Stephenson in the beginning of 20th century [1], and it was analyzed and demonstrated by D. J. Acheson [2,3]. It was again studied by J. Baillieul [4].

A pendulum at the upright position has been used in the lectures [5] and text books [6] as an example of unstable controlled objects which can be stabilized by horizontally controlling the pivot position according to the angle and the angular velocity of the pendulum. It has been also employed in many control laboratories worldwide. Not only a single but also multiple inverted pendulums up to triple ones have been stabilized by [7–9]. Such multiple pendulums are found controllable, and many advanced control technologies have been applied to evaluate the controller's performance. For example, the stabilization of a triple spherical pendulum [10] and the stabilization and transfer of a pendulum over robot hands [11] have been also successfully done. The mechanical structure of the spherical triple pendulum was concurrently designed with the controller. Use of an observer could make it possible to stabilize double inverted pendulums without measuring the angle between links [12]. A hinge actuated pendulum has been called acrobot [13,14], and has been also stabilized.

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Swinging pendulums from the pendant up to the upright position is a rather new topic which has first been studied by one of the authors including double pendulums case [15–18]. A predetermined feedforward input was applied to swing a pendulum up to the neighbourhood of the upright position, and then it is stabilized by linear feedback control. But this swing-up control was not robust. Thus more robust swing-up controls have been developed based on the vector field of a pseudo-state space, energy and artificial gravity derived by energy or similar potential function [19,20]. These control strategies need both control laws for swing-up and stabilization around the upright position. The region of attraction and the analysis of the global stabilization were studied by Zhao and Spong [21]. Instead of switching control laws for swing-up and stabilization, a nonlinear control based on State Dependent Riccati Equation (SDRE) has been also found effective for swinging-up the Furuta Pendulum shown in Fig. 1 [22].

Fig. 1. Picture of a single Furuta pendulum

The swing-up motion behaviour, however, has not been studied in detail. This paper analyzes the swing-up motion with the saturating actuator from the swing-up time. To simplify the analysis, this paper considers only the behaviour of a single pendulum. We consider the swing-up time may show the degree of stability, and try to visualize the degree using the colour map with respect to initial states and control parameters. Unstable sets of complex-value nonlinear systems with respect to initial states and parameters has been analyzed by Julia and Mandelbrot. These famous sets are called Julia set and Mandelbrot set [23,24], which are coloured using computer graphics. Taking a hint from their studies, this paper makes colour maps for the analysis of the swing-up time on the given condition for initial states and control parameters. Especially, a variable structure control presents an interesting colour map of the swing-up time.

2. Modelling of pendulum

For modeling multiple pendulums, a straight forward way is based on the projection method [25].

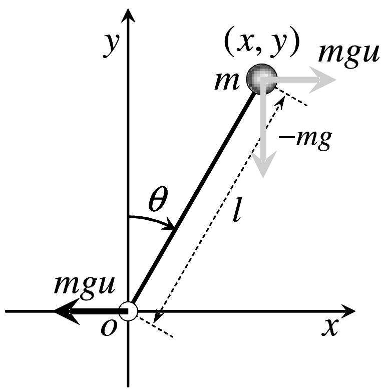

Fig. 2. Schematic figure of an ideal pendulum

The following example illustrates this idea. Let us consider an ideal pendulum in a local coordinate frame as shown in Fig. 2. A single pendulum is attached to a pivot with a free joint. The force mgu is applied to the mass by the acceleration of the moving pivot. We assume that the pivot moves left if positive input u is applied.

In the modelling, the pendulum is treated as a constrained system described by

$$M\ddot{q} = F + C^T\lambda \tag{1}$$

where

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad q = \begin{bmatrix} x \\ y \end{bmatrix}, \quad F = \begin{bmatrix} mgu \\ -mg \end{bmatrix}.$$

C is the constraint matrix representing holonomic constraints and is described below. λ is the Lagrange multiplier vector corresponding to constraint forces.

The holonomic constraint in this case is

$$x^2 + y^2 = l^2.$$

From its derivative, we obtain

$$[x\ y]\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 0 \tag{2}$$

or

$$C\dot{q} = 0. \tag{3}$$

Hence C is defined as

$$C = [x\ y]. \tag{4}$$

The generalized coordinate q can be represented by the independent coordinate θ of the reduced dimension as follows:

$$q = g(\theta) \tag{5}$$
$$g(\theta) := \begin{bmatrix} l\sin\theta \\ l\cos\theta \end{bmatrix}.$$

Then

$$\dot{q} = J\dot{\theta}$$
$$J := \frac{\partial g}{\partial \theta} = \begin{bmatrix} l\cos\theta \\ -l\sin\theta \end{bmatrix}.$$

The constraint (3) means

$$CJ = 0.$$

Using (5), the constrained system (1) can be described in the subspace of the reduced dimension.

$$J^T MJ\ddot{\theta} + J^T M\dot{J}\dot{\theta} = J^T F. \tag{6}$$

Note that the constraint force is annihilated automatically by multiplying the matrix J, because the movement of the system is projected and restricted on the constrained subspace. Furthermore, each constraint is independent, that is, C is row full rank, and thus the constraint force is given by

$$\begin{aligned}\lambda &= -(CM^{-1}C^T)^{-1}(CM^{-1}F + \dot{C}\dot{q}) \\ &= \frac{-xmgu + mgy - m\dot{x}^2 - m\dot{y}^2}{l^2}.\end{aligned} \tag{7}$$

The constraint force is a function of the pendulum length and mass, and the position and velocity of the center of gravity. This force should be taken into consideration in control system designs, for example, not to exert excessive force at the hinge.

Arranging (6), the ideal pendulum is described by the independent coordinate θ as

$$ml^2\ddot{\theta} = mgl\sin\theta + mglu\cos\theta. \tag{8}$$

This system (8) is known to be uncontrollable at the state of $\theta = \pi/2 \pmod{2\pi}$, but reachable to the upright position by proper controls. Let $g/l = 1.0$ to discuss the dynamic behaviour of the pendulum more simply. In this case, the pendulum is called a normalized pendulum, and is represented by

$$\ddot{\theta} = \sin\theta + u\cos\theta. \tag{9}$$

Let us consider several general dynamic behaviours of the normalized pendulum in the following.

3. Control strategy for swing-up pendulum

3.1. Map showing swing-up time. In this paper, our focus is on a time required for swinging-up a pendulum from the pendant to the upright-position. We call it swing-up time, and consider that the swing-up time may show the degree of stabilities. Julia and Mandelbrot have analyzed unstable sets of complex-value nonlinear systems with respect to initial states and control parameters, respectively. These famous sets are called Julia and Mandelbrot sets, which have been drawn using computer graphics. Taking a hint from their studies, this paper makes colour maps for the analysis of the swing-up time on the given condition for initial states and control parameters.

However, the proposed method to make the colour map is different from Julia and Mabdelbrot sets. In our method, a colour is decided according to the time when a trajectory started from an initial state with control parameters goes into a cylinder whose radius is small enough, as shown in Fig. 3. The radius means a tolerance of the equilibrium point. If the trajectory goes into the cylinder so fast, black colour is chosen, conversely, if the trajectory converges so slowly or does not go into the cylinder, red colour is chosen. The relation between the swing-up time and the colour is represented by a colour code bar.

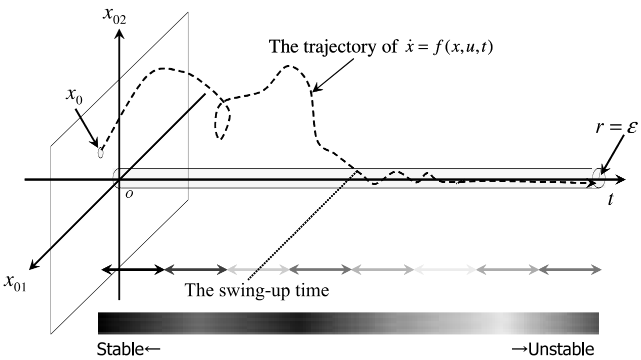

Fig. 3. Schematic figure how to make a colour map showing the swing-up time: the colour is decided according to the time when a trajectory started from the initial state goes into a cylinder showing tolerance

Finally, if we analyze an influence of the given initial state on the swing-up time, a map is made by plotting at the position of the initial state with the corresponding colour, like Julia set. In other case, that is, if we analyze an influence of the given control parameters on the swing-up time, a map is made by plotting at the position in the parameters' space with the corresponding colour, like Mandelbrot set. We will apply this analysis way to several swing-up methods, and present some interesting results.

3.2. Artificial gravity. The dynamic behaviour of pendulums has been studied for a long time, but controlling pendulums from the pendant to the upright position is a relatively new topic. Several control laws to swing-up pendulums have been studied based on energy, artificial gravity and so on. For example, the control law $u(t)$ for the system (8) by the artificial gravity is given as

$$u(t) = -2\tan\theta, \tag{10}$$

and yields

$$ml^2\ddot{\theta} = -mgl\sin\theta. \tag{11}$$

In this case, the direction of gravity is artificially changed to upward. The pendulum can be swung-up from any initial state, and also be stabilized at the upright position by adding a proper damping factor to this control. A colour map showing the swing-up time from each initial state on the map is presented in Fig. 4. The control input, however, becomes infinitely large when the pendulum is horizontal.

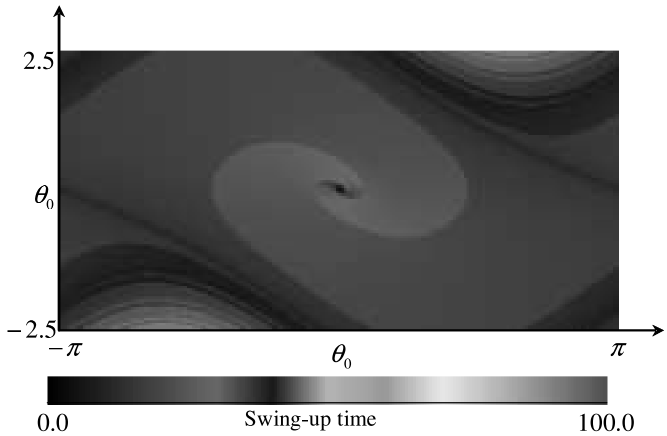

Fig. 4. A colour map showing the swing-up time from the corresponding initial state on the map by the artificial gravity

3.3. Energy based swing-up control. Let us survey a swing-up control based on energy [4]. The ideal pendulum behaviour is described in equation (8). The total energy of the pendulum including kinematic and potential is given by

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta.$$

In order to swing-up the pendulum from the pendant to the upright equilibrium position, its energy must be increased from $-mgl$ to mgl. We set $E_0 = mgl$. Now investigate the relationship between the control input u and the energy E. By simple computation,

$$\dot{E} = mglu\dot{\theta}\cos\theta$$

is easily yielded. From the derivative, the energy increases if $u\dot{\theta}\cos\theta > 0$. Hence, in order to swing-up the pendulum to the upright position, u should be chosen so that $(E - E_0)^2$ decreases, that is,

$$u = -u_{\max}\ \mathrm{sgn}((E - E_0)\dot{\theta}\cos\theta)$$

or

$$u = \begin{cases} u_{\max} & \text{if } (E - E_0)\dot{\theta}\cos\theta < 0 \\ -u_{\max} & \text{if } (E - E_0)\dot{\theta}\cos\theta > 0 \end{cases}.$$

A simulation result is shown in Fig. 5. In the simulation, we used the parameters, $m = 1.0$ [kg], $l = g = 9.8$ [m/s^2] and $u_{\max} = 1.0$, and also the initial state of $[\theta(0),\ \dot{\theta}(0)]^T = [-\pi,\ 0]^T$.

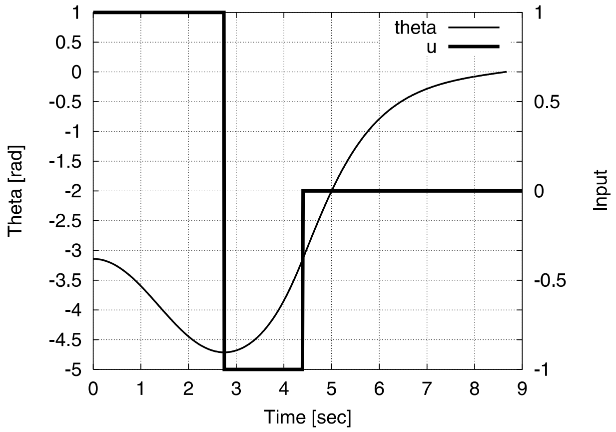

Fig. 5. A trajectory and input in simulation by the energy based swing-up control: the pendulum angle is the red solid line, and the input is blue line

From the graph, it is found that the swing started in the opposite direction to the swing-up direction toward the upright position. If the maximum amplitude of the input is small, the number of the swings may increase before the pendulum has been swung-up to the upright position. Figures 6 are colour maps showing the swing-up time from the given initial state to the upright position by the energy-based control. These figures clearly show that the swing-up time becomes longer if the maximum amplitude is small. From all initial state, the pendulum can swing-up to the upright position in the time corresponding to the colour.

4. Nonlinear control for swing-up

In order to discuss the intrinsic characteristics of swing-up control laws, let us study a simple pendulum by using colour maps. The normalized pendulum (9) can be rewritten as

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ \frac{\sin\theta}{\theta} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \cos\theta \end{bmatrix} u \tag{12}$$

where

$$x = [\theta, \dot{\theta}]^T.$$

One of effective swing-up approaches is a LQ type control law using State Dependent Riccati Equation (SDRE) [26], which considers a criterion function:

$$J = \int_0^\infty (\bar{q}_1 \sin^2\theta + \bar{q}_2\dot{\theta}^2 + \cos^2\theta u^2)dt. \tag{13}$$

The control law based on SDRE is given by

$$u = -(\cos\theta)^{-1}[f, \sqrt{\bar{q}_2 + 2f}]x \tag{14}$$

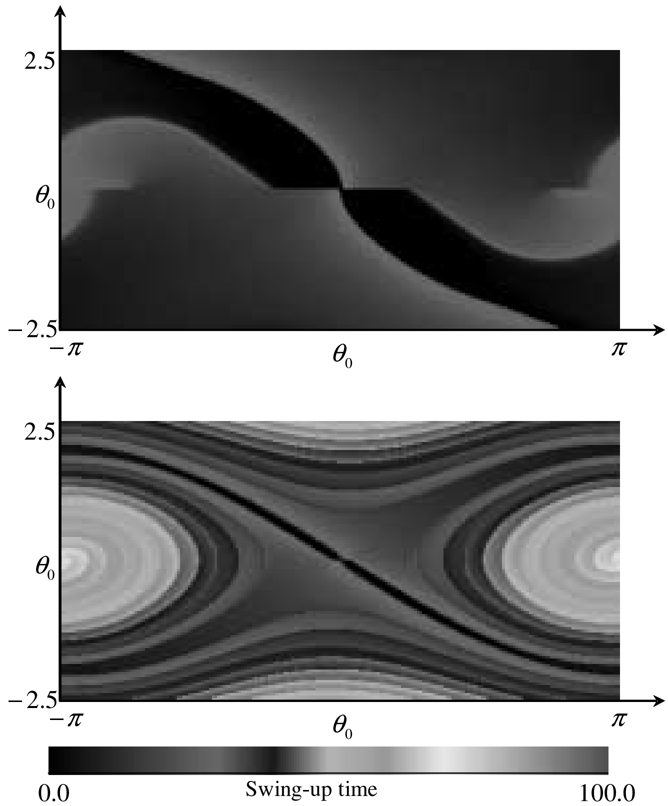

Fig. 6. The energy-based approach gives colour maps showing the swing-up time from given initial states to the upright position. In the upper map, the maximum amplitude of the input is $u_{max} = 1.0$, and in the lower map the maximum amplitude is $u_{max} = 0.1$

where

$$f = \frac{\sin\theta}{\theta} + \left|\frac{\sin\theta}{\theta}\right| \sqrt{1 + \bar{q}_1}.$$

The control law is actually given by a positive definite solution of the following SDRE:

$$\begin{bmatrix} 0 & \frac{\sin\theta}{\theta} \\ 1 & 0 \end{bmatrix} P(x) + P(x) \begin{bmatrix} 0 & 1 \\ \frac{\sin\theta}{\theta} & 0 \end{bmatrix} + \begin{bmatrix} \bar{q}_1 \left(\frac{\sin\theta}{\theta}\right)^2 & 0 \\ 0 & \bar{q}_2 \end{bmatrix} - P(x) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} P(x) = 0.$$

Then

$$2\frac{\sin\theta}{\theta} p_{12} + \bar{q}_1 \left(\frac{\sin\theta}{\theta}\right)^2 - p_{12}^2 = 0 \tag{15}$$

$$2p_{12} + \bar{q}_2 - p_{22}^2 = 0 \tag{16}$$

where p_{ij} is the (i, j) component of $P(x)$. Figures 7 show the swing-up time with the SDRE-based control law. The left-side map shows the swing-up time from the initial state $\theta = 179\pi/180$, $\dot{\theta} = 0$ using the feedback gain designed with the weights of $\bar{q}_1 = 5.0$ and $\bar{q}_2 = 5.0$. The below map shows the time for each initial state on the

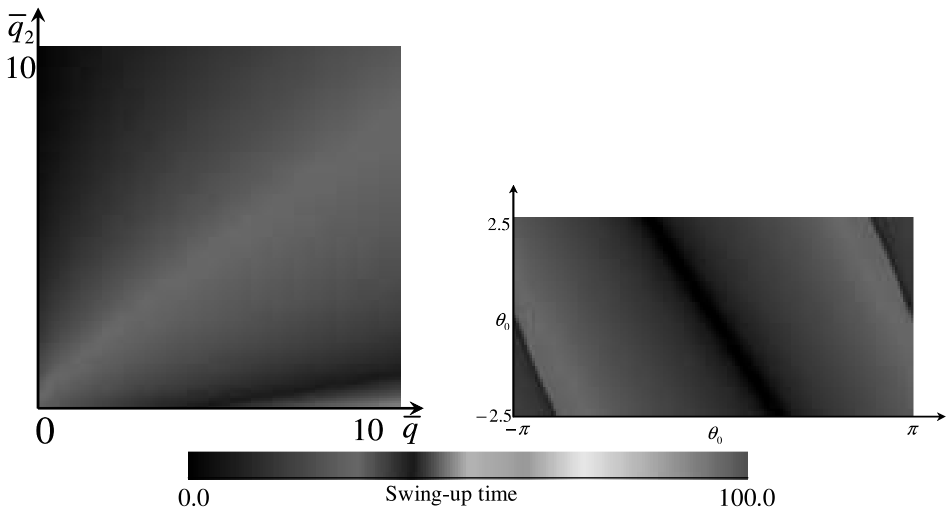

Fig. 7. The left-side map shows the swing-up time for each weight ($\bar{q}_1$, $\bar{q}_2$) from the initial state $\theta = 179\pi/180$ and $\dot{\theta} = 0$. The right-side map shows the swing-up time from each initial states on the map with the weight $\bar{q}_1 = 5$ and $\bar{q}_2 = 5$

map with the weight of $\bar{q}_1 = 5.0$ and $\bar{q}_2 = 5.0$. From Figs. 7, we find that the swing-up from almost all initial state to the upright position can be achieved by the SDRE-based control.

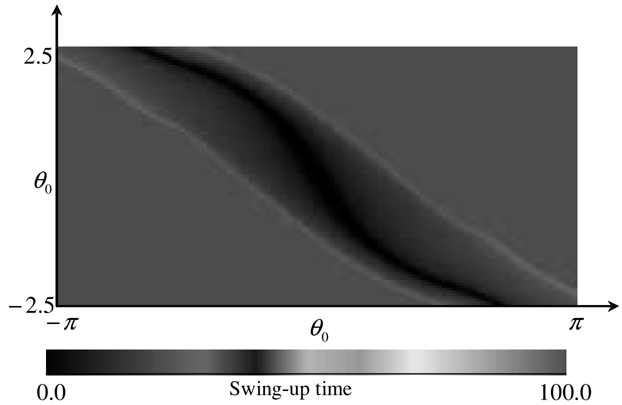

Fig. 8. A colour map showing the swing-up time from each initial state on the map by SDRE-based control law: The input limit of $K = 1.5$ and the weights of $\bar{q}_1 = 5$ and $\bar{q}_2 = 5$ are used

However, this control law also becomes infinitely large at the uncontrollable state. Such control laws include the singularity at the uncontrollable state, and then the input becomes infinitely large. Thus the behaviour with saturated input should be studied if we consider applications to real systems. We introduce the following input saturation:

$$u = \begin{cases} K & u > K \\ u & -K \leqslant u \leqslant K. \\ -K & u < -K \end{cases} \tag{17}$$

For example, setting $K = 1.5$ in Fig. 8, the colour map of the swing-up time for each initial state has been separated into two regions. The red region means the swing-up is impossible, and the other means the swing-up is possible. Even if the SDRE-based controller is used, the system cannot be stabilized from some initial states. Then, we try to analyze the case in that a simplified control law is used. The control law is obtained by keeping the structure of (14) and introducing constant coefficients f_0 and f_1, and is given by

$$u(t) = -(\cos\theta)^{-1}(\sin\theta + f_0 \sin\theta + f_1\dot{\theta}). \tag{18}$$

The closed loop system is represented by

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -f_0\frac{\sin\theta}{\theta} & -f_1 \end{bmatrix} x. \tag{19}$$

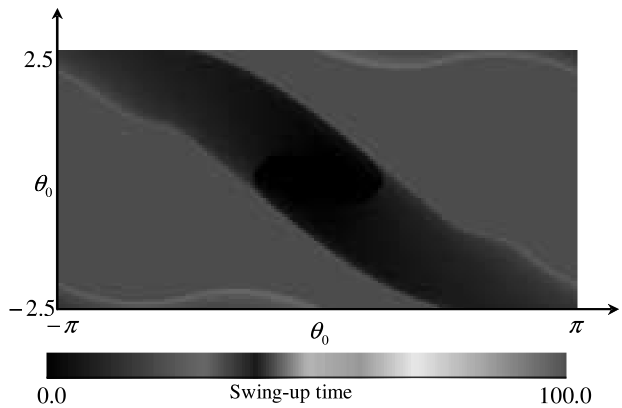

Fig. 9. Colour map showing the swing-up time from each initial state on the map using the simplified control with its feedback gains of $f_0 = \sqrt{6}$, $f_1 = \sqrt{5}$

Figure 9 is the colour map of the swing-up time in this case. To compare the result of this case with the SDRE case, the control parameters f_0 and f_1 were chosen as $f_0 = \sqrt{6}$ and $f_1 = \sqrt{5}$. From the figures, we can find that almost same result can be obtained by using the simplified control law, if the input saturation exists. Paying our attention to this interesting fact, let us consider and analyze the swing-up by a more simple control law in the following section.

5. Linear-like controller for swing-up of pendulum

In this section, a non-singular control law is used to swing-up a pendulum, and its behaviour will be analyzed. Since the angle has cyclic characteristics, a sine function is used to give a linear-like control law:

$$u = -(f_0 \sin\theta + f_1\dot{\theta}). \tag{20}$$

The normalized system (9) has a stable equilibrium point around $\theta = \pi \pmod{2\pi}$ and an unstable one around $\theta = 0 \pmod{2\pi}$ with no input. (In the following, (mod 2π) will be omitted.) Unfortunately, this control law is impossible to swing-up the pendulum from all initial states. Let us study the set of swung-up initial states.

By properly choosing the input, the system becomes stable around $\theta = 0$. Let

$$x = [\sin\theta, \dot{\theta}]^T.$$

The normalized pendulum controlled by (20) is represented as

$$\frac{d}{dt}x = \begin{bmatrix} 0 & \cos\theta \\ (1 - f_0\cos\theta) & -f_1\cos\theta \end{bmatrix} x. \tag{21}$$

Characteristics of this type of control are analyzed by making a colour map of controller parameters $(f_0,\ f_1)$ with respect to the swing-up time from the initial states $(\theta_0 = \pi + \epsilon,\ \dot{\theta}_0 = 0)$ to the upright position. If the pendulum is in the steady state at the pendant position, the pendulum does not move since the pendant position is the unstable equilibrium by the given control law. In order to avoid this condition, a small perturbation ϵ is added to the initial state, and is also added to the control input when $\theta = \pi$ and $\dot{\theta} = 0$. The result is shown in Fig. 10, where $\epsilon = -\pi/180$. The horizontal coordinate of the figure is f_0 and the vertical one is f_1. The red colour shows that it takes long time to swing-up the pendulum, and the other colours indicate the swing-up can be achieved quickly. If once the state moves outside of the non-red areas, the successive motion does not reach to the upright position, but to certain limit cycle. Therefore, the map shows reachable sets to the upright position by the swing-up time. It is an interesting characteristic that the non-red areas in the map are not connected, and stripe shapes appear in the map. From the result, note that small changes of the control parameters may cause to fail the swing-up from the given initial state to the upright position.

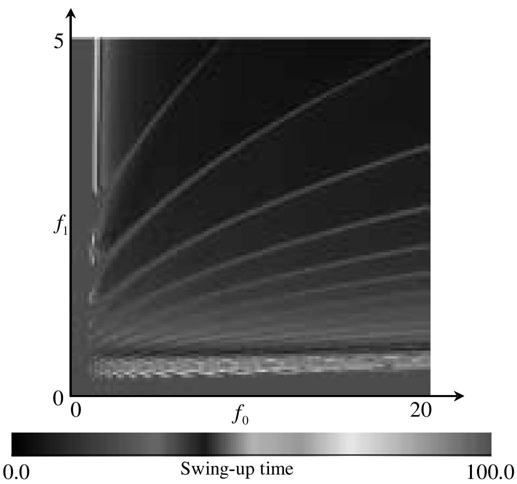

Fig. 10. Colour map of the swing-up time with respect to controller's coefficients $(f_0,\ f_1)$: The colour in the map has been decided from the colour code bar corresponding to the swing-up time from the initial state $\theta = 179\pi/180,\ \dot{\theta} = 0$ to the upright position

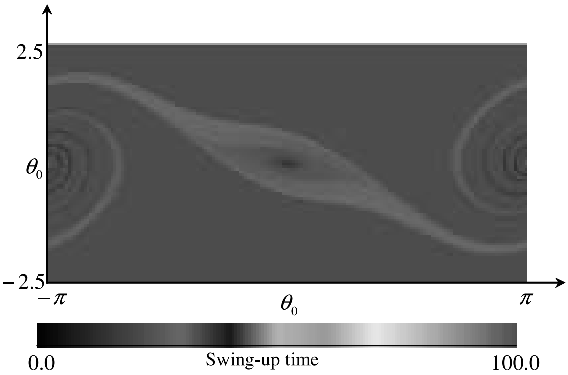

Fig. 11. Colour map of the swing-up time with respect to initial states, where the control parameters of $(f_0 = 1.5,\ f_1 = 0.5)$ are chosen

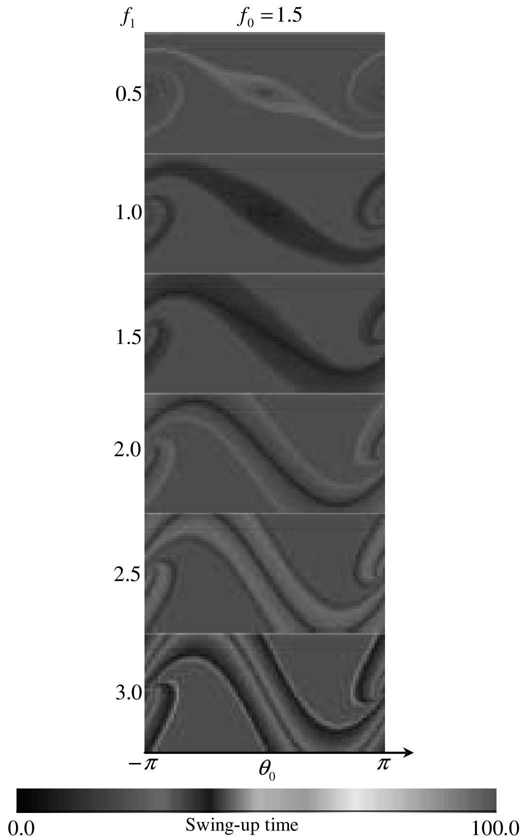

Fig. 12. Colour maps of the swing-up time with respect to initial states, which are drawn with various control parameters $f_0 = 1.5$ and $f_1 = 0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0$

Figure 10 has been drawn by fixing the initial state and changing the controller parameters. On the other hand, Figs. 11 and 12 has been drawn by fixing the control parameters and changing the initial state. θ_0 is chosen as the horizontal axis, and $\dot{\theta}_0$ is chosen as the vertical axis. The range of θ_0 is taken from $-\pi$ to π, and the range of $\dot{\theta}_0$ is taken from -2.5 to 2.5. In Figure 11, the feedback gains of $(f_0,\ f_1) = (1.5,\ 0.5)$ are used. From the figure, we find that small changes of the initial states may also cause to fail the swing-up. Moreover, in order to investigate the influence of choice of the control parameters on the map, we have drawn Figs. 12 with various parameters, that is, $f_0 = 1.5$ and $f_1 = 0.5, 1.5, 2.0, 2.5, 3.0$. The possibility of swing-up from the initial state near the pendant position is also changed as the control parameters are changed. This fact leads to Fig. 10. Especially, these maps show that the closed loop system with smaller f_1 is more sensitive to the initial state. Sets taking the coordinate of $(\sin\theta,\ \dot{\theta})$ instead of $(\theta,\ \dot{\theta})$ are shown in Fig. 13. Note that the swung-up region is transparent and the others are plotted by red colour.

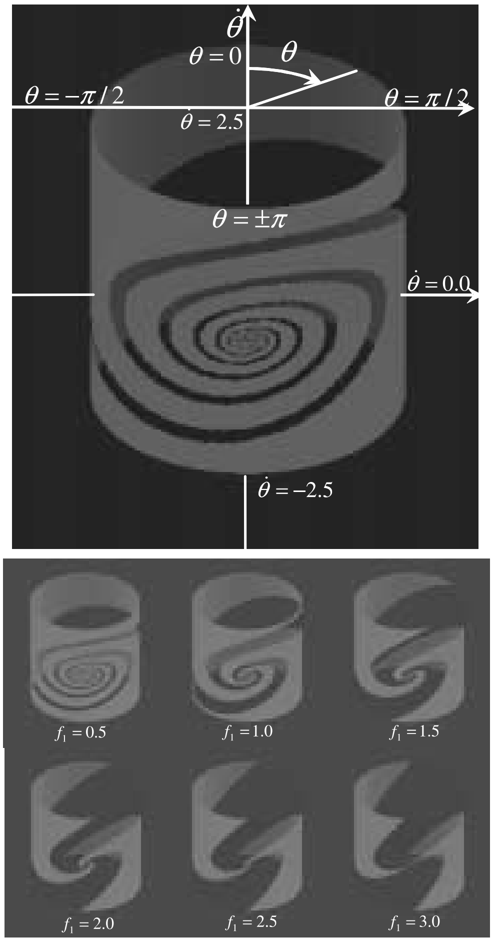

Fig. 13. Maps of the swing-up time with respect to initial states taking the cylinder coordinate $(\sin\theta,\ \dot{\theta})$: These maps are corresponding to the maps in Figs. 12, and are drawn with various control parameters $f_0 = 1.5$ and $f_1 = 0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5,\ 3.0$. However, the colours used in these maps are digitized with two colours of red and transparency

6. Sliding mode type controller for swing-up

In previous section, the control law for swing-up of the normalized pendulum with fixed control coefficients has interesting characteristics. The swing-up of the pendulum by a sliding mode type control will be also analyzed. For equation (9), a sliding mode is designed by $s = 0$ where s is given by

$$s = \dot{\theta} + a\sin\theta \tag{22}$$

with a positive constant a. Its derivative is

$$\dot{s} = \ddot{\theta} + a\cos\theta\dot{\theta} = \sin\theta + \cos\theta u + a\cos\theta\dot{\theta}. \tag{23}$$

In order to decrease $V = s^2$, a sliding mode control should be designed so as to make the derivative of V be negative. If we take

$$u = -K\,\mathrm{sgn}(s\cos\theta) \tag{24}$$

where $K > 0$, K should be infinite in order to make the derivative of V be always negative in the swing-up of the pendulum. But, it actually is not necessary to take so large K in order to swing-up from the initial states to the upright position. In this section, the swing-up time is analyzed for the choices of $(K,\ a)$. In usual sense, if the amplitude of input becomes larger, the swing-up time might become shorter. However, in this type of control, the swing-up time is not always shortened by large amplitude K of input as shown in Fig. 14.

The following interesting characteristics is found from Fig. 14. The number of preparatory swings before the swing-up has been completed depends on the control parameters K and a. However, the large K does not necessarily give the small number of the preparatory swings. In order to analyze these parameters, a colour map of the coefficients (K, a) is made, where each colour shows the swing-up time from the pendant position ($\theta = \pi, \dot{\theta} = 0$) to the upright position. As a result, a picture as a set showing the swing-up time for each parameters K and a, which is like a Fractal, can be obtained.

Given a pair of the coefficients, sets of the swing-up time from each initial state to the upright position are also analyzed. A colour map of the sets with the fixed parameter of $a = 1.4$ and $K = 1.8$ is shown in Fig. 16. This map also has interesting shape, that is, it is also like a Fractal. From the figure, note that a small difference of the initial state around which the colour changes sharply might cause a large difference of the swing-up time. In order to investigate this observation, let us pick up two initial states around the center of a spiral which has the prismatic colours in Figs. 18, and check the time response of each initial state. Figs. 18 are some parts of the zoomed colour map of Fig. 16, and Fig. 19 are the time responses. It is easily found that these observation is true, and we should pay attention to the fact in design of nonlinear controllers. The effect of the parameters can be also checked by using Figs. 17.

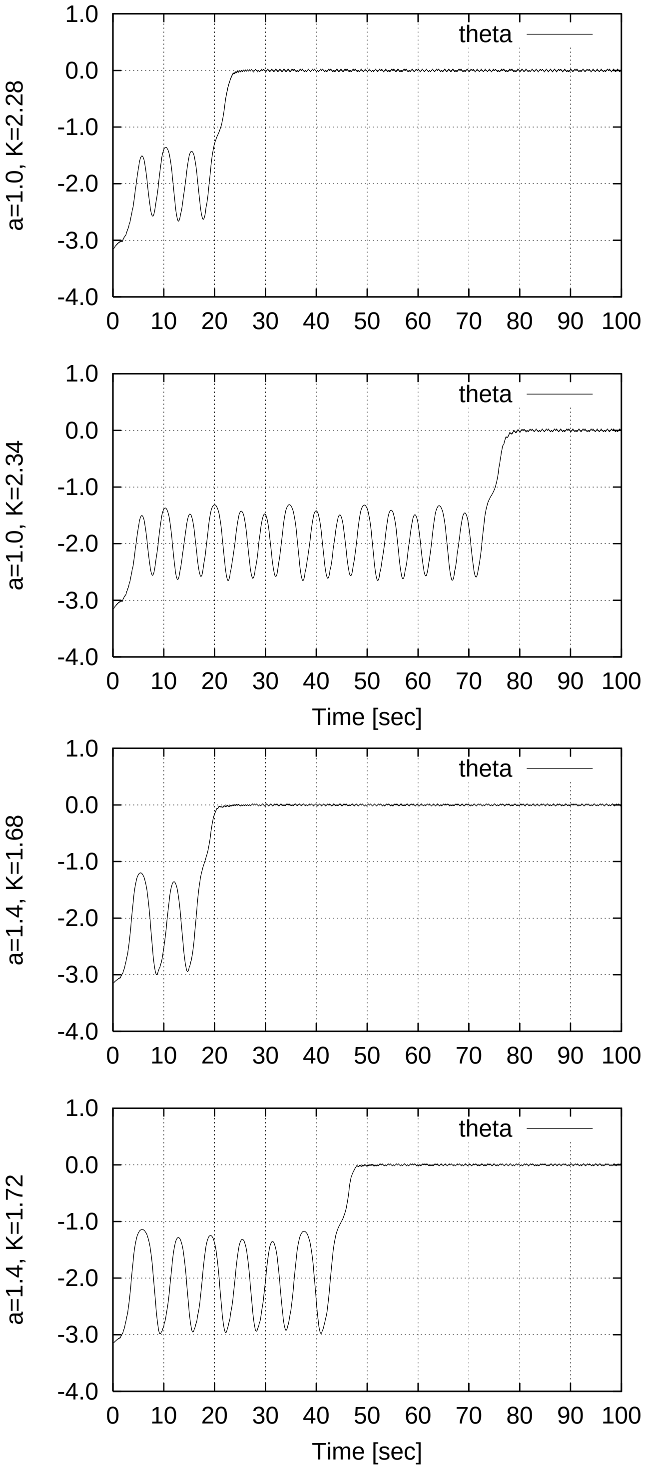

Fig. 14. The simulation results of the sliding mode type control with different control parameters: The above result is of $a = 1.0$, and the below one is of $a = 1.4$. Both results shows the fact that even if the large amplitude input is used, the swing-up time is not necessarily shortened. The number of preparatory swings is also different under different parameters a and K

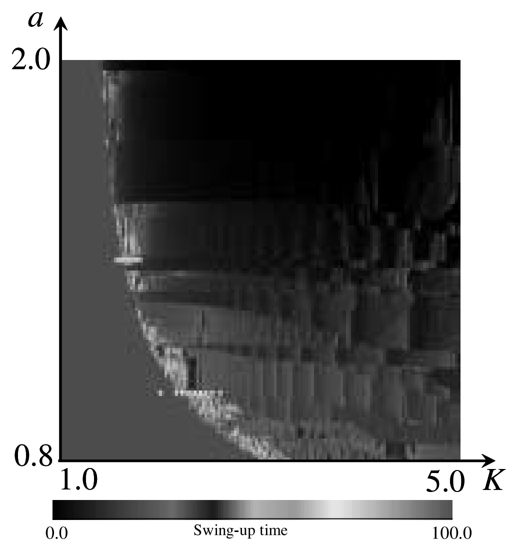

Fig. 15. A colour map showing the swing-up time by the sliding mode type controller with different parameters K and a

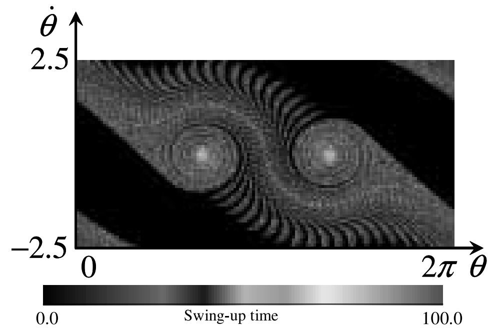

Fig. 16. A colour map showing the swing-up time by the sliding mode type controller from different initial states. The control parameters of $a = 1.4$ and $K = 1.8$ are used

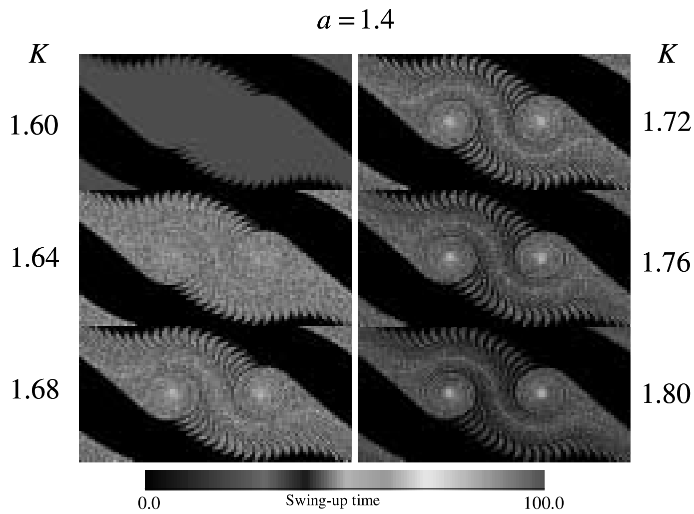

Fig. 17. Colour maps showing the swing-up time from each initial states with different control parameters of $a = 1.4$ and $K = 1.60$, 1.64, 1.68, 1.72, 1.76, 1.80

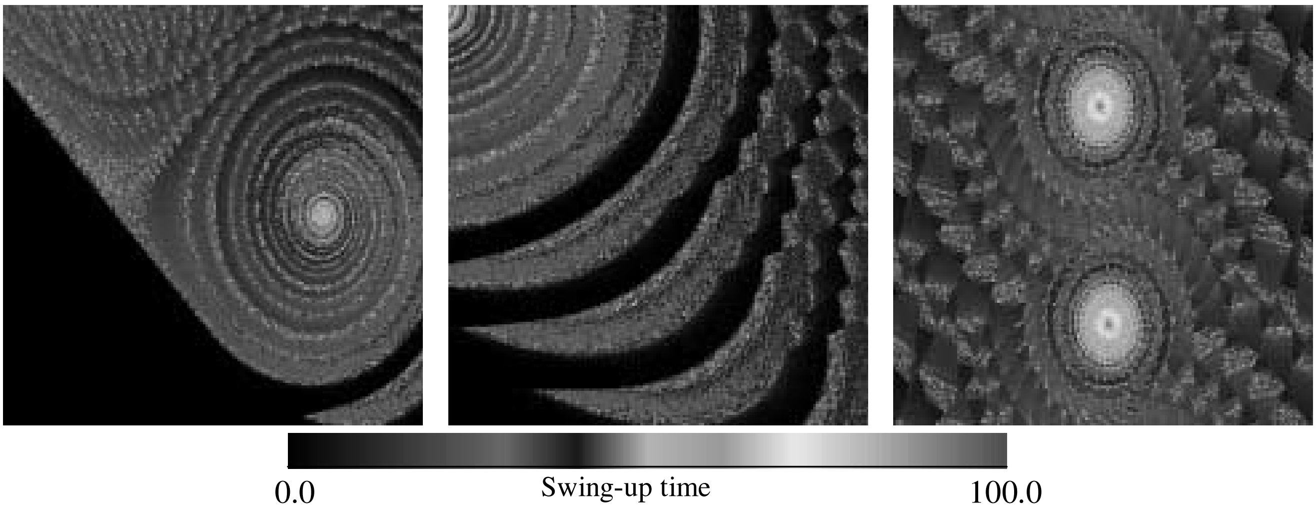

Fig. 18. Some pieces of the enlarged colour map of Fig. 16

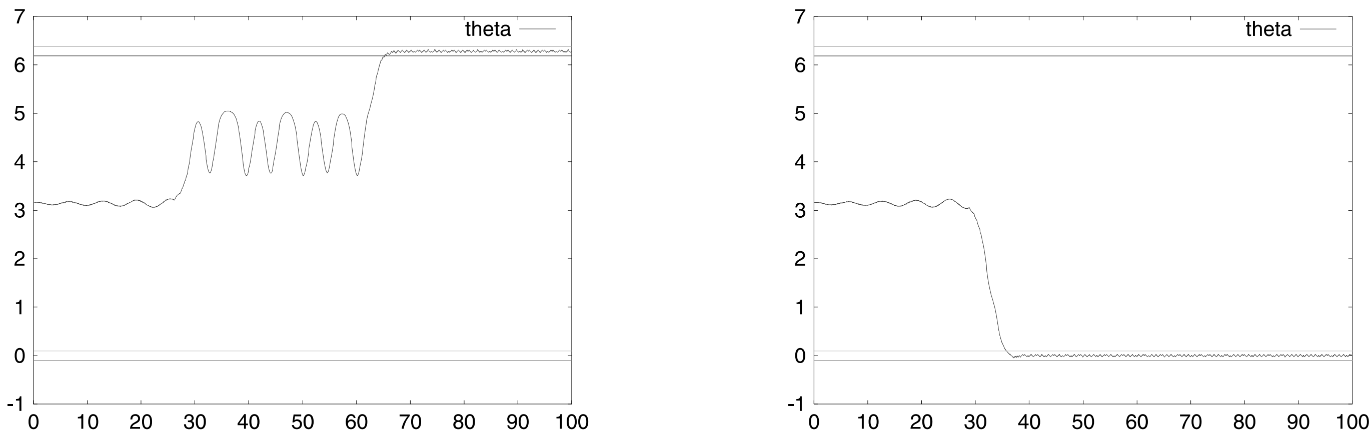

Fig. 19. Time responses of swing-up of the normalized pendulum: these two graphs show the time responses from the initial states around the center of a spiral which has prismatic colours in Fig. 18. Actually, the left-side graph is of $\theta = 3.16007$, $\dot{\theta} = 0.136364$, and the ride-side one is of $\theta = 3.16$, $\dot{\theta} = 0.13$. The control parameters of $a = 1.0$ and $K = 2.6$ are used in both graphs

If unknown parameters are introduced and an adaptive control scheme is applied, more interesting results can be obtained. Instead of (9), let us consider

$$\ddot{\theta} = c\sin\theta + \cos\theta u \tag{25}$$

where c is an unknown parameter to be estimated. The same sliding mode is considered by s = 0 where s is given by (22), and

$$V = \frac{1}{2}(s^2 + \gamma\hat{c}^2)$$

is chosen as a Lyapunov function. $\hat{c}$ is the estimated parameter of c. Then the derivative of V is

$$\begin{aligned} \dot{V} &= s\dot{s} + \gamma\hat{c}\dot{\hat{c}} \\ &= (a\dot{\theta} + u)(s\cos\theta) + \gamma(c - \hat{c})s\sin\theta, \end{aligned} \tag{26}$$

where the parameter update law is given by

$$\dot{\hat{c}} = -\frac{1}{\gamma}s\sin\theta.$$

Then, a control input is obtained by

$$u = -K\,\mathrm{sgn}(s\cos\theta).$$

We investigate the effect of the gain of input K through the following simulations under almost same conditions used in the previous sliding mode control. The simulation has been done under the different condition on K from $K = 1.0$ to $K = 10.0$ with 1.0 step. The parameter a is set as $a = 3.0$, and the initial value of $\hat{c}$ is set as $\hat{c}_0 = 0.5$. Figures 20 show the results of the simulation, which are colour maps showing the swing-up time from each initial state by adaptive sliding mode controller with each K, like Julia-like sets. As it is seen in Figs. 20, the stability is not always improved even if K is increased. These interesting phenomena can be seen in the control pendulum.

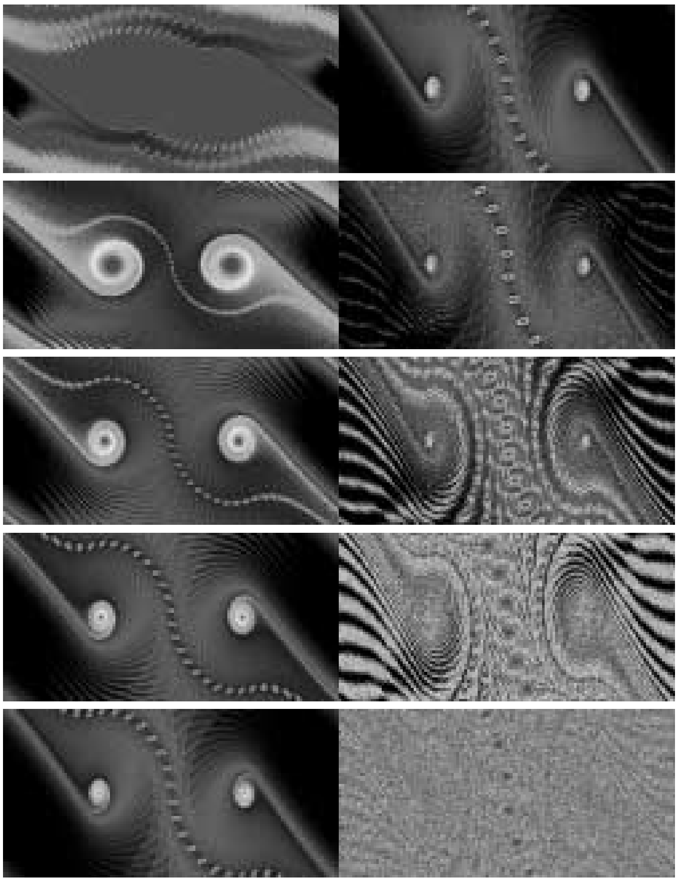

Fig. 20. Colour maps of initial states showing the swing-up time by adaptive sliding mode type controller with $a = 3.0$ and several $K = 1.0,\ 2.0,\ 3.0,\ 4.0,\ 5.0$ in the left column, $K = 6.0,\ 7.0,\ 8.0,\ 9.0,\ 10.0$ in the right column

7. Conclusion

The several swing-up controls of a single pendulum from the pendant to the upright position have been studied. We first restudied the traditional control methods from the swing-up time point of view. Nonlinear controls for swing-up of pendulum based on State Dependent Riccati Equation and sliding mode have been also presented. In these studies, the colour maps have been made and utilized for analyses of the swing-up time. The swing-up time of controlled pendulums for controller parameters and initial states has been studied by using the maps. These maps are interesting and complicated similar to Mandelbrot and Julia sets in Fractals.

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