BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 52, No. 1, 2004

Constrained controllability of semilinear systems with multiple delays in control

J. KLAMKA*

Institute of Control Engineering, Silesian Technical University, 16 Akademicka St., 44-100 Gliwice, Poland

Abstract. In the present paper finite-dimensional, stationary dynamical control systems described by semilinear ordinary differential state equations with multiple point delays in control are considered. Infinite-dimensional semilinear stationary dynamical control systems with single point delay in the control are also discussed. Using a generalized open mapping theorem, sufficient conditions for constrained local relative controllability are formulated and proved. It is generally assumed, that the values of admissible controls are in a convex and closed cone with vertex at zero. Some remarks and comments on the existing results for controllability of nonlinear dynamical systems are also presented.

Keywords: controllability, nonlinear control systems, semilinear control systems, constrained controls, delayed control systems.

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory [1–4]. This is a qualitative property of dynamical control systems and is of particular importance in control theory. Systematic study of controllability was started at the beginning of sixties, when the theory of controllability based on the description in the form of state space for both time-invariant and time-varying linear control systems was worked out.

Roughly speaking, controllability generally means, that it is possible to steer dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. In the literature there are many different definitions of controllability, which strongly depend on class of dynamical control systems [1–3, 5, 6].

In recent years various controllability problems for different types of nonlinear dynamical systems have been considered in many publications and monographs. The extensive list of these publications can be found for example in the monograph [3] or in the survey paper [4]. However, it should be stressed, that the most literature in this direction has been mainly concerned with controllability problems for finite-dimensional nonlinear dynamical systems with unconstrained controls and without delays [1, 5–8] or for linear infinite-dimensional dynamical systems with constrained controls and without delays [2, 7, 9].

In this paper, we shall consider constrained local relative controllability problems for finite-dimensional stationary semilinear dynamical systems with multiple point delays in the control described by ordinary differential state equations. Moreover, infinite-dimensional semilinear stationary dynamical control systems with single point delay in the control are also discussed. Let us recall, that semilinear dynamical control systems contain linear and pure nonlinear parts in the differential state equations [6, 10]. More precisely, we shall formulate and prove sufficient conditions for constrained local relative controllability in a prescribed time interval for semilinear dynamical systems with multiple point delays in the control which nonlinear term is continuously differentiable near the origin. It is generally assumed that the values of admissible controls are in a given convex and closed cone with vertex at zero, or in a cone with nonempty interior. Proofs of the main results are mainly based on the so called generalized open mapping theorem presented in the paper [11].

Roughly speaking, it will be proved that under suitable assumptions constrained global relative controllability of a linear associated approximated dynamical system implies constrained local relative controllability near the origin of the original semilinear abstract dynamical system. This is a generalization to constrained controllability case of some previous results concerning controllability of linear dynamical systems with multiple point delays in the control and with unconstrained controls [2, 3, 8].

Finally, is should be mentioned, that other different controllability problems both for linear and nonlinear dynamical control systems have been also considered in the papers in the Refs. 1, 5–7, 9.

2. System description

In this paper we study the semilinear stationary finite-dimensional dynamical control system with multiple point delays in the control described by the following ordinary differential state equation

$$\dot{x}(t) = Ax(t) + F(x(t)) + \sum_{j=0}^{j=M} B_j u(t - h_j) \quad \text{for } t \in [0,T],\, T > h \tag{1}$$

with zero initial conditions:

$$x(0) = 0 \quad u(t) = 0 \quad \text{for} \quad t \in [-h, 0) \tag{2}$$

where the state $x(t) \in R^n = X$ and the control $u(t) \in R^m = U$, A is $n \times n$ dimensional constant matrix B_j, $j = 0, 1, 2, \ldots, M$ are $n \times m$ dimensional constant ma-

* e-mail: jklamka@ia.polsl.gliwice.pl

trices, $0 = h_0 < h_1 < \ldots < h_j < \ldots << h_m = h$ are constant delays. Moreover, let us assume that the nonlinear mapping $F : X \to X$ is continuously differentiable near the origin and such that $F(0) = 0$.

In practice admissible controls are always required to satisfy certain additional constraints. Generally, for arbitrary control constraints it is rather very difficult to give easily computable criteria for constrained controllability. However, for some special cases of the constraints it is possible to formulate and prove simple algebraic constrained controllability conditions. Therefore, we assume that the set of values of controls $U_c \subset U$ is a given closed and convex cone with nonempty interior and vertex at zero. Then the set of admissible controls for the dynamical control system (1) has the following form $U_{\text{ad}} = L_\infty([0,T], U_c)$.

Then for a given admissible control $u(t)$ there exists a unique solution $x(t;u)$ for $t \in [0,T]$, of the state equation (1) with zero initial condition (2) described by the integral formula [6, 10]

$$x(t;u) = \int_0^t S(t-s)(F(x(s;u)) + \sum_{j=1}^{j=M} B_j u(t-h_j))ds \quad (3)$$

where the semigroup $S(t) = \exp(At)$ is $n \times n$ transition matrix for the linear part of the semilinear control system (1).

For the semilinear dynamical system with single point delay in the control (1), it is possible to define many different concepts of controllability. In the sequel we shall focus our attention on the so called constrained relative controllability in the time interval $[0,T]$. In order to do that, first of all let us introduce the notion of the attainable set at time $T > 0$ from zero initial conditions (2), denoted by $K_T(U_c)$ and defined as follows [2, 3, 5]

$$K_T(U_c) = \{x \in X : x = x(T,u), u(t) \in U_c \text{ for a.e.} t \in [0,T] \quad (4)$$

where $x(t,u)t > 0$ is the unique solution of the equation (1) with zero initial conditions (2) and a given control u. Under the assumptions stated on the nonlinear term F such solution always exists [6, 10].

Now, using the concept of the attainable set, let us recall the well known (see e.g. [2, 3, 8]) definitions of constrained relative controllability in $[0,T]$ for dynamical system (1).

Definition 2.1. The dynamical system (1) is said to be U_c-locally relative controllable in $[0,T]$ if the attainable set $K_T(U_c)$ contains a neighbourhood of zero in the space X.

Definition 2.2. The dynamical system (1) is said to be U_c-globally relative controllable in $[0,T]$ if $K_T(U_c) = X$.

3. Preliminaries

In this section, we shall introduce certain notations and present some important facts from the general theory of nonlinear operators.

Let U and X be given spaces and $g(u) : U \to X$ be a mapping continuously differentiable near the origin 0 of U. Let us suppose for convenience that $g(0) = 0$. It is well known from the implicit-function theorem (see e.g. [11]) that, if the derivative $Dg(0) : U \to X$ maps the space U onto the whole space X, then the nonlinear map g transforms a neighbourhood of zero in the space U onto some neighbourhood of zero in the space X.

Now, let us consider the more general case when the domain of the nonlinear operator g is Ω, an open subset of U containing 0. Let U_c denote a closed and convex cone in U with vertex at 0.

In the sequel, we shall use for controllability investigations some property of the nonlinear mapping g which is a consequence of a generalized open-mapping theorem [11]. This result seems to be widely known, but for the sake of completeness we shall present it here, though without proof and in a slightly less general form sufficient for our purpose.

Lemma 3.1 [11]. Let X, U, U_c, and Ω be as described above. Let $g : \Omega \to X$ be a nonlinear mapping and suppose that on Ω nonlinear mapping g has derivative Dg, which is continuous at 0. Moreover, suppose that $g(0) = 0$ and assume that linear map $Dg(0)$ maps U_c onto the whole space X. Then there exist neighbourhoods $N_0 \subset X$ about $0 \in X$ and $M_0 \subset \Omega$ about $0 \in U$ such that the nonlinear equation $x = g(u)$ has, for each $x \in N_0$, at least one solution $u \in M_0 \cap U_c$, where $M_0 \cap U_c$ is a so called conical neighbourhood of zero in the space U.

Lemma 3.2. Let $D_u x$ denotes derivative of x with respect to u. Moreover, if $x(t;u)$ is continuously differentiable with respect to its u argument, we have for $v \in L_\infty([0,T], U) D_u x(t;u)(v) = z(t,u,v)$ where the mapping $t \to z(t,u,v)$ is the solution of the linear ordinary equation

$$\dot{z}(t) = Az(t) + D_x(F(x;u))z(t) + \sum_{j=0}^{j=M} B_j v(t-h_j) \quad (5)$$

with zero initial conditions

$$z(0,u,v) = 0 \quad \text{and} \quad v(t) = 0 \quad \text{for} \quad t \in [-h, 0).$$

P r o o f . Using formula (3) and the well known differentiability results we have

$$D_u x(t;u) = \int_0^t D_u(S(t-s)(F(x(t;u)) + \sum_{j=0}^{j=M} B_j u(t-h_j)))ds$$
$$= \int_0^t S(t-s) D_u(F(x(t;u)) + \sum_{j=0}^{j=M} B_j u(t-h_j))ds$$

$$= \int_0^t S(t-s) D_x F(x(t;u)) D_u x(t;u) ds$$

$$+ \int_0^t S(t-s) (\sum_{j=0}^{j=M} B_j u(t-h_j)) ds. \quad (6)$$

Differentiating equality (6) with respect to the time variable t, we have

$$(d/dt) D_u x(t;u) v =$$

$$= D_x F(x(t;u)) D_u x(t;u) v + \sum_{j=0}^{j=M} B_j u(t-hj) v$$

$$+ \int_0^t (d/dt) S(t-s) \sum_{j=0}^{j=M} B_j u(t-hj)) ds v$$

$$+ \int_0^t (d/dt) S(t-s) D_x F(x(s;u)) D_u x(s;u) ds v. \quad (7)$$

Therefore, since by assumption $S(t)$ is a differentiable semigroup then $(d/dt)S(t-s) = AS(t-s)$ and we have

$$\dot{z}(t) = D_x F(x(t;u)) z(t) + \left(\int_0^t AS(t-s) \sum_{j=0}^{j=M} B_j v(t-h_j)) ds \right) + \left(\int_0^t AS(t-s) D_x F(x(s;u)) z(s) ds \right) \quad (8)$$

On the other hand solution of the equation (5) has the following integral form

$$z(t) = \int_0^t S(t-s) \cdot \left(D_x F(x(s;u)) z(s) + \sum_{j=0}^{j=M} B_j v(t-h_j) \right) ds \quad (9)$$

Therefore, differential equation (8) can be expressed as follows

$$\dot{z}(t) = Az(t) + D_x F(x(t;u)) z(t) + \sum_{j=0}^{j=M} B_j v(t-h_j).$$

Hence Lemma 3.2 follows.

4. Controllability conditions

In this section we shall study constrained local relative controllability in $[0,T]$ for semilinear dynamical system (1) using the associated linear dynamical system with multiple point delays in the control

$$\dot{z}(t) = Cx(t) + \sum_{j=0}^{j=M} B_j u(t-h_j) \quad (10)$$

for $t \in [0,T]$ with zero initial conditions $z(0) = 0$, $u(t) = 0$, for $t \in [-h, 0)$ where

$$C = A + D_x F(0). \quad (11)$$

The main result is the following sufficient condition for constrained local relative controllability of the semilinear dynamical system (1).

THEOREM 4.1. Suppose that

(i) $F(0) = 0$,

(ii) $U_c \subset U$ is a closed and convex cone with vertex at zero,

(iii) The associated linear control system with multiple point delays in the control (10) is U_c-globally relative controllable in $[0,T]$.

Then the semilinear dynamical control system with multiple point delays in the control (1) is U_c-locally relative controllable in $[0,T]$.

Proof. Let us define for the nonlinear dynamical system (5) a nonlinear map

$$g : L_\infty([0,T], U_c) \to X \text{ by } g(u) = x(T,u).$$

Similarly, for the associated linear dynamical system (10), we define a linear map $H : L_\infty([0,T], U_c) \to X$ by $Hv = z(T,v)$.

By the assumption (iii) the linear dynamical system (10) is U_c-globally relative controllable in $[0,T]$. Therefore, by the Definition 2.2 the linear operator H is surjective, i.e. it maps cone of admissible controls U_{ad} onto the whole space X. Furthermore, by Lemma 3.2 we have that $Dg(0) = H$.

Since U_c is a closed and convex cone, then the cone of admissible controls $U_{\text{ad}} = L_\infty([0,T], U_c)$ is also a closed and convex cone in the function space $L_\infty([0,T], U)$. Therefore, the nonlinear map g satisfies all the assumptions of the generalized open mapping theorem stated in the Lemma 3.1. Hence, the nonlinear map g transforms a conical neighbourhood of zero in the cone of admissible controls U_{ad} onto some neighbourhood of zero in the state space X. This is by Definition 2.1 equivalent to the U_c-local relative controllability in $[0,T]$ of the semilinear dynamical control system (1). Hence, our theorem follows.

In practical applications of the Theorem 4.1, the most difficult problem is to verify the assumption (iii) about constrained global controllability of the linear dynamical system (10) (see, e.g. [1–3, 7, 9] for more details). In order to avoid this serious disadvantage, we may use the following Theorem.

THEOREM 4.2 [2, 8, 9]. Suppose the set U_c is a cone with vertex at zero and a nonempty interior in the space R^m.

Then the associated linear dynamical control system (10) is U_c-globally relatively controllable in $[0,T]$ if and only if

(1) it is relative controllable without any constraints, i.e. rank$[B_0, B_1, \ldots, B_M, CB_0, CB_1, \ldots, CB_M, C^2B_0, C^2B_1, \ldots, C^2B_M, \ldots, C^{n-1}B_0, C^{n-1}B_1, \ldots, C^{n-1}B_M] = n$,

(2) there is no real eigenvector $v \in R^n$ of the matrix C^{tr} satisfying

$$v^{\mathrm{tr}} Bu \leqslant 0 \quad \text{for all} \quad u \in U_c.$$

Let us observe, that for a special case when $T < h_1$, relative controllability problem in $[0, T]$ for dynamical system with delays in control may be reduced to the well known standard controllability problem for dynamical control system without delays in the control [2].

COROLLARY 4.1 [2, 8]. Suppose that $T < h_1$ and the assumptions of Theorem 4.1 are satisfied.

Then the associated linear dynamical control system (10) is U_c-globally controllable in $[0, T]$ if and only if it is controllable without any constraints, i.e.

$$\mathrm{rank}[B_0, CB_0, C^2B_0, \ldots, C^{n-1}B_0] = n,$$

and there is no real eigenvector $v \in R^n$ of the matrix C^{tr} satisfying $v^{\mathrm{tr}} Bu \leqslant 0$ for all $u \in U_c$.

It should be pointed out that for the single input associated linear dynamical control system (10), i.e. for the case $m = 1$, Theorem 4.2 reduces to the following Corollary.

COROLLARY 4.2 [2, 8, 9]. Suppose that $m = 1$ and $U_c = R^+$.

Then the associated linear dynamical control system (10) is U_c-globally relative controllable in $[0, T]$ if and only if it is relative controllable without any constraints, i.e.
$\mathrm{rank}[B_0,\ B_1,\ \ldots,\ B_{M.},\ CB_0,\ CB_1,\ \ldots,\ CB_{M.},\ C^2B_0,\ C^2B_1,\ \ldots,\ C^2B_{M.},\ \ldots,\ C^{n-1}B_0,\ C^{n-1}B_1,\ \ldots,\ C^{n-1}B_M] = n$,
and matrix C has only complex eigenvalues.

COROLLARY 4.3 [2, 8, 9]. Suppose that $m = 1$, $M = 1$, and $U_c = R^+$.

Then the associated linear dynamical control system (10) is U_c-globally controllable in $[0, T]$ if and only if it is controllable without any constraints, i.e.

$$\mathrm{rank}[B_0, B_1, CB_0, CB_1, C^2B_0, C^2B_1, \ldots \\ \ldots, C^{n-1}B_0, C^{n1}B_1] = n,$$

and matrix C has only complex eigenvalues.

COROLLARY 4.4 [2, 8, 9]. Suppose that $T < h_1$, $m = 1$ and $U_c = R^+$.

Then the associated linear dynamical control system (10) is U_c-globally controllable in $[0, T]$ if and only if it is controllable without any constraints, i.e.

$$\mathrm{rank}[B_0, CB_0, C^2B_0, \ldots, C^{n-1}B_0] = n,$$

and matrix C has only complex eigenvalues.

5. Infinite-dimensional systems with delay in control

In this section we study the semilinear infinite-dimensional abstract control system with single constant point delay in the control

$$x'(t) = Ax(t) + F(x(t)) + B_0u(t) + B_1u(t-h) \\ \text{for} t \in [0, T] \quad (12)$$

with zero initial conditions:

$$x(0) = 0 \quad u(t) = 0 \quad \text{for} t \in [-h, 0) \quad (13)$$

where the state $x(t)$ takes values in a real Banach space X and the control $u(t)$ is in another real Banach space U.

Let us assume that the linear generally unbounded operator A generates a strongly differentiable semigroup $S(t)$ on X for $t \geqslant 0$ and B_0, B_1 are linear bounded operators from U to X. Assume that the nonlinear mapping $F : X \to X$ is continuously Frechet differentiable near the origin and such that $F(0) = 0$.

Let $U_c \subset U$ be a closed convex cone with nonempty interior and vertex at zero. The set of admissible controls for the dynamical system (1) is $U_{\mathrm{ad}} = L_\infty([0, T], U_c)$.

Then for a given admissible control $u(t)$ there exists a unique mild solution $x(t; u)$ of the equation (1) with zero initial condition (2) described by the integral formula [18–20, 29].

$$x(t; u) = \int_0^t S(t-s)(F(x(s; u)) + B_0u(s) \\ + B_1u(t-h))ds. \quad (14)$$

For the semilinear abstract dynamical system with delay in the control (1), it is possible to define many different concepts of controllability. In the sequel we shall focus our attention on the so called constrained exact relative controllability in the time interval $[0, T]$, $T > h$. In order to do that, first of all let us introduce the notion of the attainable set at time $T > 0$ from zero initial state $x(0) = 0$, denoted by $K_T(U_c)$ and defined as follows

$$K_T(U_c) = \{x \in X : x = x(T, u), u(t) \in U_c, \\ \text{for } t \in [0, T]\} \quad (15)$$

where $x(t, u)$, $t > 0$ is the unique solution of the equation (1) with zero initial conditions (2) and given control u. Under the assumptions stated on the nonlinear term F such solution always exists [18–20, 29].

Now, using the concept of the attainable set given above, let us recall the well known [13, 23, 28] definitions of constrained exact relative controllability for dynamical system (1).

DEFINITION 5.1. The dynamical system (1) is said to be U_c-exactly locally relative controllable in $[0, T]$ if the attainable set $K_T(U_c)$ contains a neighbourhood of zero in the space X.

For the finite-dimensional case i.e., when $X = R^n$, we may omit the word “exact” in the Definition 5.1 since in this case exact local relative controllability is equivalent to approximate local relative controllability, (see Definition 2.1).

Definition 5.2. The dynamical system (1) is said to be U_c-exactly globally relative controllable in $[0,T]$ if $K_T(U_c) = X$.

Similarly as in the previous case, for $X = R^n$, we may omit the word "exact" in Definition 5.2 since in this case exact global relative controllability is equivalent to approximate global relative controllability, (see Definition 2.2).

6. Exact controllability conditions

In this section we shall study constrained exact relative controllability for the infinite-dimensional dynamical control system (12) using the associated linear infinite-dimensional dynamical system

$$z'(t) = Cz(t) + B_0u(t) + B_1u(t-h) \quad \text{for } t \in [0,T] \quad (16)$$

with zero initial conditions

$$z(0) = 0, \quad u(t) = 0 \quad \text{for } t \in [-h, 0)$$

where the linear operator C is given by the following equality

$$C = A + D_xF(0). \quad (17)$$

The main result is the following sufficient condition for constrained exact relative controllability of the semilinear dynamical system (12).

Theorem 6.1. Suppose that

(i) $F(0) = 0$,

(ii) $U_c \subset U$ is a closed convex cone with vertex at zero,

(iii) The linear system (16) is U_c-exactly globally relative controllable in $[0,T]$.

Then the semilinear dynamical system (12) is U_c-exactly locally relative controllable in $[0,T]$.

P r o o f . Proof of Theorem 6.1 is almost the same as the proof of Theorem 4.1, and hence will be omitted.

Corollary 6.1. Suppose that the assumptions (i) and (ii) of the Theorem 6.1 are satisfied, and that the cone U_c has nonempty interior in the space U.

Then the nonlinear dynamical system (1) is U_c-exactly locally relative controllable in $[0,T]$ if the associated linear dynamical system (16) is U-exactly globally relative controllable, i.e. without any constraints and

$$\ker(sI - C^*) \cap (BU_c)^0 = \{0\} \text{ for every } s \in R. \quad (18)$$

P r o o f . In the proof we shall use the condition for constrained exact global relative controllability of the associated linear dynamical system (16). If the cone U_c has nonempty interior in the space U and the condition (18) is satisfied, then U-exact global controllability in $[0,T]$ of the associated linear dynamical system (16) implies its U_c-exact global relative controllability in $[0,T]$ [30]. Therefore, assumption (iii) of the Theorem 6.1 is satisfied and our Corollary follows.

The conditions for exact global relative controllability of linear infinite-dimensional dynamical systems which are needed in the Theorem 6.1 and Corollary 6.1 are known to be quite a strong requirement [2, 6, 10]. Exact global relative controllability for linear dynamical systems does not hold, for example if the corresponding semigroup of solution linear operators is compact, or if the operator B is compact, see, e.g. [2, 5–7, 10] for more details. This situation includes, for example distributed parameter dynamical systems described by partial differential equations, or infinite-dimensional dynamical systems with finite-dimensional controls [2].

However, it should be stressed that exact relative controllability may occur in certain linear subspaces of the state space for the case of dynamical control systems described by linear partial differential equations of hyperbolic type with a suitably chosen state space [5, 6].

Similarly, linear retarded functional differential systems may be exactly relative controllable in an appropriate defined state space [7, 10, 11]. In this cases the following corollary may be useful in constrained exact relative controllability problems.

Corollary 6.2. Let $X_e \subset X$ be a linear subspace in which the linear dynamical control system (16) is U_c-exactly globally relative controllable in $[0,T]$. Moreover, suppose that the assumptions (i) and (ii) of the Theorem 4.1 are satisfied.

Then the semilinear dynamical system (12) is U_c-exactly locally relative controllable in $[0,T]$ in the linear subspace X_e.

P r o o f . Since all the assumptions of the Theorem 6.1 are satisfied, then Corollary 6.2 follows directly from the Theorem 6.1.

Now, let us consider the special case when $T < h$. In this situation exact relative controllability is equivalent to exact controllability and thus we have the following corollary.

Corollary 6.3 [2, 8]. Suppose that $T < h$ and the assumptions of Theorem 6.1 are satisfied.

Then the associated linear dynamical control system (16) is U_c-globally exactly controllable in $[0,T]$ if and only if it is globally exactly controllable without any constraints.

7. Example

Let us consider the following simple illustrative example. Let the semilinear finite-dimensional dynamical control system defined on a given time interval $[0,T]$, $T > h$, has the following form

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t) + u(t-h_1) \\ x_2(t) &= \sin x_1(t) + u(t-h_2) \end{aligned} \quad (19)$$

Therfeore, we have

$$n = 2,\ m = 1,\ M = 2,\ 0 = h_0 < h_1 < h_2 = h,$$
$$x(t) = (x_1(t), x_2(t))^{\text{tr}} \in R^2 = X,\ U = R,$$

and using the notations given in the previous sections matrices A and B and the nonlinear mapping F have the following form

$$A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$F(x) = F(x_1, x_2) = \begin{bmatrix} 0 \\ \sin x_1 \end{bmatrix}.$$

Moreover, let the cone of values of controls $U_c = R^+$, and the set of admissible controls
$U_{\mathrm{ad}} = L_\infty([0,T], R^+)$.

Hence, we have

$$F(0) = F(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathrm{D_x}F(x) = \begin{bmatrix} 0 & 0 \\ \cos x_1 & 0 \end{bmatrix}$$

$$\mathrm{D_x}F(0) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathrm{C} = \mathrm{A} + \mathrm{D_x}F(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, the matrix C has only complex eigenvalues and

$$\operatorname{rank}[B_0, B_1, B_2, CB_0, CB_1, CB_2] = \operatorname{rank} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} = 2 = n.$$

Hence, both assumptions of the Corollary 4.2 are satisfied and therefore, the associated linear dynamical control system (10) is R^+-globally controllable in a given time interval $[0,T]$. Then, all the assumptions stated in the Theorem 4.1 are also satisfied and thus the semilinear stationary dynamical control systems (19) is R^+-locally controllable in $[0,T]$. However, it should be mentioned, that since

$$\operatorname{rank}[B_0, B_1, CB_0, CB_1] = \operatorname{rank} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 1 < n = 2$$

then the semilinear finite-dimensional stationary dynamical control system (19) may be not relative controllable in the interval $[0,T]$, for $T < h_2$, even for unconstrained controls.

8. Concluding remarks

In the present paper sufficient conditions for constrained local relative controllability near the origin for semilinear finite-dimensional stationary dynamical control systems with multiple point delays in the control have been formulated and proved.

Moreover, sufficient conditions for constrained local exact relative controllability near the origin for semilinear infinite-dimensional stationary dynamical control systems with single point delay in the control have been also presented.

In the proofs of the main results generalized open mapping theorem [11] has been extensively used. The relative controllability conditions given in the present paper extend to the case of constrained relative controllability of finite and infinite-dimensional semilinear stationary dynamical control systems, the results published in [1, 2, 7] and [8] for unconstrained nonlinear stationary control systems.

The method presented in the present paper is in fact quite general and covers wide class of semilinear dynamical control systems. Therefore, similar constrained relative controllability results may be derived for more general class of semilinear dynamical control systems.

For example, it seems, that it is possible to extend sufficient constrained relative controllability conditions given in the previous sections for infinite-dimensional semilinear dynamical control systems with distributed delay in the control or with multiple point delays in the state variables.

Moreover, quite similar nonlinear analysis methods can be used to solve controllability problems for the discrete-time abstract infinite-dimensional semilinear control systems with multiple delays in the control and state variables.

References

[1] E. N. Chukwu and S. M. Lenhart, "Controllability questions for nonlinear systems in abstract spaces", *J. Opt. Theory and Appl.* 68(3), 437–462 (1991).

[2] J. Klamka, *Controllability of Dynamical Systems*, Dordrecht: Kluwer Academic Publishers, 1991.

[3] J. Klamka, "Controllability of dynamical systems — a survey", *Archives of Control Sciences* 2(3/4), 281–307 (1993).

[4] G. Peichl and W. Schappacher, "Constrained controllability in Banach spaces", *SIAM J. Contr. and Optimization* 24(6), 1261–1275 (1986).

[5] T. I. Seidman, "Invariance of the reachable set under nonlinear perturbations", *SIAM J. Contr. and Optimization* 25(5), 1173–1191 (1987).

[6] H. X. Zhou, "Controllability properties of linear and semilinear abstract control systems", *SIAM J. Contr. and Optimization* 22(3), 405–422 (1984).

[7] N. Fujii and Y. Sakawa, "Controllability for nonlinear differential equations in Banach space", *Automat. Contr. Theory and Appl.* 2(2), 44–46 (1974).

[8] J. Klamka, "Constrained controllability of nonlinear systems", *J. Math. Anal. and Applications* 201(2), 365–374 (1996).

[9] N. K. Son, "A unified approach to constrained approximate controllability for the heat equations and the retarded equations", *J. Math. Anal. and Applications* 150(1), 1–19 (1990).

[10] K. Naito, "Controllability of semilinear control systems dominated by the linear part", *SIAM J. Contr. and Optimization* 25(3), 715–722 (1987).

[11] S. M. Robinson, "Stability theory for systems of inequalities. Part II. Differentiable nonlinear systems", *SIAM J. Numerical Anal.* 13(4), 497–513 (1976).