Frequency optimization based on semi-analytical and "exact" numerical differentiation methods

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Abstract. In this paper exact sensitivity and optimization of free vibrating systems is considered. Finite element formulation of natural frequencies problem is presented. It is known that natural frequencies of free vibrations can be treated as eigenvalues of some generalized eigenproblem. Reduction of the eigenvalue system algorithm is shown. The sensitivity of eigenvalues with respect to design parameter is presented. The stiffened plate and double-disk rotor shaft system are taken as an example. The goal of optimization is to move natural frequencies as far as possible from resonance frequency. The sensitivities of objective function are calculated by three different methods: Finite Difference Method (FDM), semi-analytical method and the Direct Differentiation Method (DDM) [1, 2, 3]. The perturbations in FDM and semi-analytical method are calculated. FDM, semi-analytical method and DDM results are compared. Next the optimization is performed with the objective to move natural frequencies as far as possible from resonance frequency. First example is a stiffened plate with two beams, second a double-disk rotor shaft system. Design parameters are height of stiffeners and diameters of parts of shaft respectively.

1 INTRODUCTION

In this paper the optimization of stiffened plate based on sensitivity analysis is considered. The objective is to avoid resonance, which can cause excessive stresses leading to prematural fatigue cracks for instance. Sensitivities of eigenvalues with respect to design variables are calculated by the Finite Difference Method (FDM), semi-analytical method and the Direct Differentiation Method (DDM) [1, 2, 3]. The perturbations in FDM and semi-analytical method are calculated. FDM, semi-analytical method and DDM results are compared. Next the optimization is performed with the objective to move natural frequencies as far as possible from resonance frequency. First example is a stiffened plate with two beams, second a double-disk rotor shaft system. Design parameters are height of stiffeners and diameters of parts of shaft respectively.
2 NATURAL FREQUENCY PROBLEM

Let's consider natural frequency problem without dumping [4, 5]

\[ M \ddot{u} + Ku = 0 \]  \hspace{1cm} (1)

where, \( M \) and \( K \) are mass and stiffness matrices respectively, \( u \) is displacement vector, \( \dot{\cdot} \) denotes time derivative.

Assumption is made that, solution of the problem is

\[ u(\tau) = \Phi \sin[\omega(\tau - t_0)] \]  \hspace{1cm} (2)

where \( \omega \) is frequency and \( \Phi \) is \( N \)-dimensional vector of normalized vibration amplitude, \( \tau \) is time. After differentiation of \( u(\tau) \), substitution into Eqn (1) and transformation to global system we obtain eigenproblem equation

\[ (K - \Lambda M) \Phi = 0 \]  \hspace{1cm} (3)

where \( \Lambda \) is diagonal matrix of square natural frequencies \( \omega \). \( \Phi \) is global matrix of eigenvectors Eqn (4).

\[ \Lambda = \begin{bmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_N^2 \end{bmatrix} \]

\[ \Phi = \begin{bmatrix} \Phi_1(1) & \Phi_1(2) & \cdots & \Phi_1(N) \\ \Phi_2(1) & \Phi_2(2) & \cdots & \Phi_2(N) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_N(1) & \Phi_N(2) & \cdots & \Phi_N(N) \end{bmatrix} \]  \hspace{1cm} (4)

This is a general linear eigenvalue or characteristic value problem, for non-zero solutions the determinant

\[ |K - \Lambda M| = 0 \]  \hspace{1cm} (5)

When stiffness and mass matrices are symmetric and positive defined (all the diagonals of the triangular factors are positive and diagonals are dominating) the determinant gives \( N \) real and different values of \( \omega_i^2 \).

The solution of Eqn (5) does not allows to determine actual values of \( \Phi \). Such vectors are known as the normal modes of the system or eigenvectors and are made unique by normalizing. \( \Phi \) is orthogonal matrix, so that

\[ \Phi^T M \Phi = I \]  \hspace{1cm} (6)

calls M-orthonormality and

\[ \Phi^T K \Phi = \Lambda \]  \hspace{1cm} (7)

calls K-orthogonality where, \( I \) is identity matrix.

If we make the system dependent on design parameter vector \( h \), eigenvalues and corresponding eigenvectors are also dependent on design parameters vector, we can then rewrite Eqn (3) as

\[ (K(h) - \Lambda(h)M(h)) \Phi(h) = 0 \]  \hspace{1cm} (8)

3 REDUCTION OF THE EIGEN SYSTEM

Determination of all pairs eigenvalue-eigenvector is difficult. Furthermore number of real eigenpairs of the real system is infinite. We want to know generally a only small number of lowest dominating eigenfrequencies with largest normal modes. It is possible to simplify calculations by reducing the size of eigenproblem.

To obtain reduced eigenproblem we assume that the \( N \) unknown vectors \( \Phi_i \) can be represented by \( m << N \) vectors \( \mathbf{t}_j \) with corresponding factors \( x_j \) [5]. It can be written, as:

\[ \tilde{\Phi} = \mathbf{t}_1 x_1 + \mathbf{t}_2 x_2 + \ldots + \mathbf{t}_m x_m = \mathbf{T}x \]  \hspace{1cm} (9)

which substituted into Eqn (3), gives

\[ KTx - \Lambda M Tx = 0 \]  \hspace{1cm} (10)
the vectors

The choice of the trial vectors is not arbitrary. If side is equal zero (see Eqn (8)).

\[ K^*x - \Lambda^*M^*x = 0 \] (11)

where

\[ K^* = T^KT, \quad M^* = T^MT \] (12)

Now \( \Lambda^* \) consist of eigenvalues of reduced system. The choice of the trial vectors is not arbitrary. If the vectors \( t \) will be chosen as eigenvectors of the original matrix the system will be diagonal and \( \Lambda^* \) will be equal to \( m \) first values of \( \Lambda \). Program Feap, which is used by author to get solutions, finds vectors \( t \) in iterative subspace algorithm.

4 EIGENVALUE SENSITIVITY

If the mass and stiffness matrices are symmetric and continuously differentiable with respect to design variable vector \( h \) and if any eigenvalue \( \omega_i \) is not repeated, then eigenvalues and corresponding eigenvectors are also continuously differentiable with respect to design variables [5, 4]. After differentiation of Eqn (8) we obtain:

\[
\frac{\partial K(h)}{\partial h} \tilde{\Phi}(h) + K(h) \frac{\partial \tilde{\Phi}(h)}{\partial h} - \frac{\partial \Lambda(h)}{\partial h} M(h) \tilde{\Phi}(h) - \Lambda(h) \frac{\partial M(h)}{\partial h} \tilde{\Phi}(h) - \Lambda(h) M(h) \frac{\partial \tilde{\Phi}(h)}{\partial h} = 0
\] (13)

By leaving \( \frac{\partial \Lambda(h)}{\partial h} \) factor on the left-hand side, transferring others factors on the right-hand side and putting in order right-hand side factors, we obtain Eqn (14). The second term on right-hand side is equal zero (see Eqn (8)).

\[
\frac{\partial \Lambda(h)}{\partial h} M(h) \tilde{\Phi}(h) = \left( \frac{\partial K(h)}{\partial h} - \Lambda(h) \frac{\partial M(h)}{\partial h} \right) \tilde{\Phi}(h)
\] (14)

Multiplication of both sides of this equation by \( \Phi^T \) gives:

\[
\frac{\partial \Lambda(h)}{\partial h} \Phi^T(h) M(h) \Phi(h) = \Phi^T(h) \left( \frac{\partial K(h)}{\partial h} - \Lambda(h) \frac{\partial M(h)}{\partial h} \right) \Phi(h)
\] (15)

Using the orthonormality condition Eqn (6), we obtain the sensitivity equation Eqn (16).

\[
\frac{\partial \Lambda(h)}{\partial h} = \Phi^T(h) \left( \frac{\partial K(h)}{\partial h} - \Lambda(h) \frac{\partial M(h)}{\partial h} \right) \Phi(h)
\] (16)

Calculations of natural frequencies of the system give all terms in the reduced eigenproblem Eqn (11) and (12). After substitution Eqn (9) into general sensitivity formula Eqn (16), we obtain:

\[
\frac{\partial \Lambda(h)}{\partial h} = x^T(h) T^T(h) \frac{\partial K(h)}{\partial h} T(h) x(h) - \Lambda(h) x^T(h) T^T(h) \frac{\partial M(h)}{\partial h} T(h) x(h)
\] (17)

Rearranging the right-hand side of Eqn (17) gives eigenvalue sensitivity formula for reduced eigenproblem:

\[
\left( \frac{\partial \Lambda(h)}{\partial h} \right)^* = x^T(h) \left[ \left( \frac{\partial K(h)}{\partial h} \right)^* - \Lambda(h) \left( \frac{\partial M(h)}{\partial h} \right)^* \right] x(h)
\] (18)

where

\[
\left( \frac{\partial K(h)}{\partial h} \right)^* = T^T(h) \frac{\partial K(h)}{\partial h} T(h)
\]

\[
\left( \frac{\partial M(h)}{\partial h} \right)^* = T^T(h) \frac{\partial M(h)}{\partial h} T(h)
\] (19)

and \( \left( \frac{\partial \Lambda(h)}{\partial h} \right)^* \) contains the first \( m \) values of \( \left( \frac{\partial \Lambda(h)}{\partial h} \right)^* \).
5 OPTIMIZATION ALGORITHM

The objective of optimization is to move the natural eigenfrequency from the neighborhood of the assumed forced frequency in order to avoid resonance. This may prolong fatigue life of the structure. The forced frequency $f_b$ is related to eigenvalue $\lambda_b$ by formula:

$$\lambda_b = (2\pi f_b)^2$$  \hspace{1cm} (20)

Optimization problem is formulated as follows:

$$c(h) = \max [|\lambda - \lambda_b|] = \min [-|\lambda - \lambda_b|], \quad \lambda \in \Lambda$$  \hspace{1cm} (21)

with constraints:

$$h_{\min} \leq h_1, h_2, ..., h_n \leq h_{\max}$$

$$|K(h) - \Lambda(h)M(h)| = 0$$  \hspace{1cm} (22)

where $\lambda$ is closest natural eigenvalue of stiffened plate.

Sensitivities of the objective function are calculated using three different methods: Finite Difference Method (FDM), semi-analytical method and Direct Differentiation Method (DDM).

The first method (FDM) estimates sensitivity by comparing values of $c(h)$ and $c(h + \Delta h_i)$ by formula:

$$\frac{\partial c(h)}{\partial h_i} \approx \frac{c(h + \Delta h_i) - c(h)}{\Delta h_i}$$  \hspace{1cm} (23)

Value of $\Delta h$ must be determined separately for each step and each design variable of optimization on the basis of perturbation chart. An example of perturbation chart is shown in Figure 2. That makes Finite Difference Method (FDM) not efficient. FDM can be applied only for smooth objective functions (sign of $\frac{\partial c(h)}{\partial h_i}$ must be constant in large range of $h$).

In sensitivity calculations of objective function by semi analytical method and DDM, differentiation of objective function (Eqn. (21)) with respect to design parameter $h_i$ is necessary. We obtain:

$$\frac{\partial c(h)}{\partial h_i} = \begin{cases} \frac{\partial \lambda(h)}{\partial h_i} & \text{for } \lambda < \lambda_b \\ -\frac{\partial \lambda(h)}{\partial h_i} & \text{for } \lambda > \lambda_b \end{cases}$$  \hspace{1cm} (24)

In DDM values of $\frac{\partial \lambda(h)}{\partial h_i}$ are calculated by formula Eqn. (18). If computation of $\left(\frac{\partial K(h)}{\partial h}\right)^*$ and $\left(\frac{\partial M(h)}{\partial h}\right)^*$ are difficult or impossible we can use so called semi-analytical method and calculate these values from formula:

$$\left(\frac{\partial K(h)}{\partial h_i}\right)^* \approx \frac{(K(h+\Delta h_i))^*-(K(h))^*}{\Delta h_i}$$

$$\left(\frac{\partial M(h)}{\partial h_i}\right)^* \approx \frac{(M(h+\Delta h_i))^*-(M(h))^*}{\Delta h_i}$$  \hspace{1cm} (25)

In Direct Differentiation Method all quantities: $\frac{\partial \lambda(h)}{\partial h_i}$, $\left(\frac{\partial K(h)}{\partial h}\right)^*$ and $\left(\frac{\partial M(h)}{\partial h}\right)^*$ are solved accurately. This method is most precise and consume less processor time. We need to solve an eigenproblem and next calculate sensitivities. In FDM we have to solve an eigenproblem at least two times (or more, if number of design parameters is greater than 1).

6 NUMERICAL EXAMPLES

6.1 The Stiffened Plate

The stiffened plate shown in Figure 1 is clamped on all boundaries. The material data are as follows: $E = 2.1 \cdot 10^{11} Pa$, Poisson’s ratio is 0.3 and density $\rho = 7850 \, kg/m^3$. Thickness of plate is equal 2 mm. Two beams with rectangular cross section represents stiffening system of the plate. The height of stiffeners ($h_1$ and $h_2$ respectively) are chosen as design variables. Lower limit of stiffeners height 15 mm is chosen.
due to physical limitations of plate deflection. Upper limit depends on the sum of heights of both stiffeners. This sum must be less or equal 60 mm due to assumed limit of weight of the system (see Eqn. (22)). Width of the stiffeners is \( b = 5 \) mm. Initial value of stiffeners height is taken as 25 mm. Forced frequencies \( f_b = 50 \) Hz and \( f_b = 140 \) Hz are considered.

The objective function may be rewritten from Eqn. (21) as:

\[
c(h) = \max \left| \lambda - \lambda_b \right| = \min \left[ -|\lambda - \lambda_b| \right],
\]

\( \lambda \in \Lambda \)

with constraints:

\[
\begin{align*}
h_{\min} &= 15 \leq h_1, h_2 \leq h_{\max} = 45 \\
h_1 + h_2 &\leq 60 \\
|K(h) - \Lambda(h)M(h)| &= 0
\end{align*}
\]

where \( \lambda \) is closest natural eigenvalue of stiffened plate.

Optimization performed with forced frequency 50 Hz is presented in Figure 4. Numbers near columns denotes consecutive optimization steps. Optimum values of design parameters is reached after eight steps. These values are: \( h_1 = 15 \) mm and \( h_2 = 45 \) mm and corresponds to objective function value \(-141.96 \) Hz. Perturbation charts in Finite Difference Method and semi-analytical

Figure 1: Initial geometry of the structure.

Figure 2: Perturbation graph of \( dh \) (FDM), forced frequency 50 Hz.

Figure 3: Perturbation graph of \( dh \) (semi-analytical method), forced frequency 50 Hz.

Figure 4: Optimization path in design space, forced frequency 50 Hz. Consecutive optimization steps are denoted by numbers.
method are similar with any value of initial $h_1$ and $h_2$. Some calculated perturbations charts are presented in Figure 2 and Figure 3. We observe in all charts of perturbation large range of $\Delta h$ where value of $\frac{\Delta c(h)}{\Delta h}$ is stable. Value of $\Delta h$ is chosen as $10^{-5}$. Sensitivities of objective function with respect to design parameters $h_1$ and $h_2$ are shown in Fig. 5 and Figure 6 respectively.

Figure 5: Normalized sensitivity of objective function with respect to $h_1$ for consecutive minimization steps, forced frequency 50 Hz.

Figure 6: Normalized sensitivity objective function with respect to $h_2$ for consecutive minimization steps, forced frequency 50 Hz.

Next optimization with forced frequency 200 Hz is performed, $h = [h_1, h_2]$. Optimization path in design space is presented in Figure 7. Process of optimization ends after six steps and corresponds to minimum value of objective function $-45.12$ Hz. Optimum values of stiffeners heights are 19 mm and 17 mm for $h_1$ and $h_2$ respectively.

Figure 7: Optimization path in design space, forced frequency 200 Hz. Consecutive optimization steps are denoted by numbers.

Figure 8: Normalized sensitivity of objective function with respect to $h_1$ (DDM) for consecutive minimization steps, forced frequency 200 Hz.

Sensitivities of objective function with respect to the height of the first ($h_1$) and the second ($h_2$) stiffener are presented in Figure 8 and Figure 9 respectively. These sensitivities are also calculated by three different methods: FDM, semi-analytical method and DDM. Obtained sensitivities have the same character but values are not in the same order. That gives a reason normalized values of sensitivities. There is quantitative but not qualitative difference between sensitivities.

In order to check what is the influence of the optimization on the fatigue life of the structure, elementary load is applied. Time dependent pressure is applied on the surface of the structure.
It is changing sinusoidally form $-1 \, kN/m^2$ to $1 \, kN/m^2$. The maximum stress value in structure is verified.

First, the frequency of the pressure is equal $50 \, Hz$. Calculations are performed for structure before optimization ($h_1 = h_2 = 25 \, mm$) and after optimization ($h_1 = 15 \, mm$, $h_2 = 45 \, mm$). Maximum stress values in structure with forced load with frequency $50 \, Hz$ is presented in Figure 10. We observe that maximum stress and its amplitude in structure after optimization decrease, and the mean value is smaller. Small difference between maximum stress before and after optimization we explain by small improvement (in percentage) of the objective function. This secondary effect of optimization prolongs structure life.

Next calculations with sinusoidal pressure with frequency equal $200 \, Hz$ and amplitude $1 \, kN/m^2$ are done. In Figure 11 maximum stress values in structure before optimization ($h_1 = h_2 = 25 \, mm$) and after optimization ($h_1 = 19 \, mm$, $h_2 = 17 \, mm$) are presented. We see that, maximum stresses after optimization are much smaller than stresses before optimization in spite of the fact, that the stiffeners cross section was smaller. This can be seen as secondary effect of frequency optimization. It will prolong fatigue strength (es-
especially high-cycles fatigue) of the structure.
In order to define real fatigue life of the structure it is necessary to determine that by more precise fatigue analysis.

6.2 The Double-Disk Rotor Shaft System

The shaft system presented on Figure 12 is considered. Dimensions are as follows: diameters, $\phi_1 = 90 \text{ mm}$, $\phi_2 = \phi_3 = 100 \text{ mm}$, $\phi_4 = 110 \text{ mm}$, $\phi_5 = \phi_6 = 130 \text{ mm}$, $\phi_7 = 130 \text{ mm}$, $\phi_8 = 130 \text{ mm}$, $\phi_9 = 130 \text{ mm}$, $\phi_{10} = \phi_{11} = 130 \text{ mm}$, $\phi_{12} = 110 \text{ mm}$, $\phi_{13} = \phi_{14} = 100 \text{ mm}$, $\phi_{15} = 152 \text{ mm}$ and length of parts, $l_1 = 370 \text{ mm}$, $l_2 = 50 \text{ mm}$, $l_3 = 80 \text{ mm}$, $l_4 = 220 \text{ mm}$, $l_5 = 100 \text{ mm}$, $l_6 = 100 \text{ mm}$, $l_7 = 200 \text{ mm}$, $l_8 = 200 \text{ mm}$, $l_9 = 200 \text{ mm}$, $l_{10} = 100 \text{ mm}$, $l_{11} = 100 \text{ mm}$, $l_{12} = 220 \text{ mm}$, $l_{13} = 80 \text{ mm}$, $l_1 = 155 \text{ mm}$, $l_1 = 145 \text{ mm}$ respectively.

Material of the shaft is steel with Young modulus $E = 2.1 \cdot 10^{11} \text{ Pa}$, Poisson’s ratio is 0.3 and density $\rho = 7850 \text{ kg/m}^3$. Both rigid disks are the same and have parameters: mass $m_1 = m_2 = 200 \text{ kg}$ and diametral mass moment of inertia $J_1 = J_2 = 3.176 \text{ kg/m}^3$. Both bearings are the same too, stiffnesses are: $k_{xx} = 3.15 \cdot 10^7 \text{ Pa}$, $k_{yy} = -3.15 \cdot 10^7 \text{ Pa}$, $k_{xy} = 1.3 \cdot 10^8 \text{ Pa}$ and $k_{yx} = -1.3 \cdot 10^8 \text{ Pa}$. The design parameters are diameters of parts number 7, 8 and 9. Initial values are $\phi_7 = \phi_8 = \phi_9 = 130 \text{ mm}$. Those diameters may vary between value of 110 mm and 150 mm. Lower limit restricts maximal stress value in those parts, upper limit is assumed due to maximum weight of shaft system. The objective function may be rewritten from Eqn. 21 and 22 as:

$$c(h) = \max [\|\lambda - \lambda_b\|] = \min [\|\lambda - \lambda_b\|], \quad \lambda \in \Lambda$$

(28)

with constraints:

$$110 \leq \phi_7, \phi_8, \phi_9 \leq 150$$

(29)

There are two cases. In case (a) rotation speed of shaft is 3000 rot/min, that gives oscillating forced load with frequency 50 Hz. This case is typically in electrical generators. Case (b) considers with rotating speed of shaft system 8400 rot/min, which gives oscillating forced load with frequency 140 Hz.

Let’s consider case (a). Consecutive optimization steps are presented on Figure 13. Sensitivities are calculated by Direct Differentiation Method. Sensitivity charts with respect to $\phi_7$, $\phi_8$ and $\phi_9$ are presented on Figure 14. Final values of design parameters are: $\phi_7 = 115 \text{ mm}$,
Figure 12: Initial and final geometries of double-disk rotor shaft system.

Figure 14: Normalized sensitivities of objective function with respect to $\phi_i \ (i = 7, 8, 9)$ for consecutive minimization steps, case (a) forced frequency 50 Hz.

$\phi_8 = 118 \text{ mm}$ and $\phi_9 = 139 \text{ mm}$. We can observe on Figure 13 that objective function was improved only by about 2 Hz. One can say that such results are not very satisfactory. Improvement is too small in relation to expected outcome. Such bad effect is caused by situation, when frequency 50 Hz is very close and greater than the first eigenfrequency. By varying diameters of parts 7, 8 and 9 we can not move first eigenfrequency lower. This bad situation might by corrected by choosing a new initial values of $\phi_7$, $\phi_8$ and $\phi_9$ in such way that first eigenfrequency will be greater than 50 Hz.

Case (b) corresponds to oscillating forced load with frequency 140 Hz. Initial values of design parameters are the same as in case (a) and equal $\phi_7 = \phi_8 = \phi_9 = 130 \text{ mm}$. All sensitivities are calculated analytically by Direct Differentiation Method. Consecutive optimization steps are presented on Figure 15. Charts of sensitivity of objective function with respect to $\phi_7$, $\phi_8$ and $\phi_9$ are
Double-disk rotor shaft system

Objective function with forced frequency 140 Hz

-47
-45
-43
-41
-39
-37
-35
1 2 3 4 5 6 7 8 9 10 11 12
optimization’s steps

Figure 15: Optimization path in design space, case (b) forced frequency 140 Hz.

Normalized sensitivities obtained by DDM with forced frequency 140 Hz

-6.0E-01
-4.0E-01
-2.0E-01
0.0E+00
2.0E-01
4.0E-01
6.0E-01
8.0E-01
1.0E+00
1.2E+00
1 2 3 4 5 6 7 8 9 10 11
optimization’s step

Figure 16: Normalized sensitivities of objective function with respect to $\phi_i$ ($i = 7, 8, 9$) for consecutive minimization steps, case (b) forced frequency 140 Hz.

6.3 Future Work

Presented approach of the resonance optimization may be applied in hydraulic cylinders to improve fatigue strength. This task is investigated by authors inside current research project. Hydraulic cylinders situated near motors, pneumatic hammers or other vibration source may fail due to premature fatigue crack. Some critical areas of possible fatigue cracks are shown in Figure 17. Figure 18 shows deformed shape of selected part of the hydraulic cylinder under pressure equal 10 MPa. More informations reader could find in [7]. Optimization with objective to move eigenfrequency as far as possible from constant forced frequency can possibly prolong working time of the cylinder.
7 CONCLUSIONS

1. Present work deals with optimization performed on the basis of exact sensitivity.

2. Optimization with objective to avoid resonance allows to eliminate undesirable vibration, noise etc.

3. In specific structures optimization with the objective to avoid resonance can give additional advantage: reduction of the weight of the construction.

4. Resonance elimination may increase fatigue strength of the structure.

References


