Fatigue analysis of the cracked rotor by means of the one- and three-dimensional dynamical model

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ABSTRACT

In the paper the structural one-dimensional hybrid dynamical model of the entire vibrating rotor-shaft system and the three-dimensional finite element model of its cracked shaft zone were applied for a fatigue life prediction of the machine faulty segment under coupled bending-torsional-axial vibrations. The steady-state dynamic response amplitudes, obtained by means of the one-dimensional model of the system, have been used for the three-dimensional model as an input data for determination of maximal stresses and stress intensity factors at the crack tip. These quantities together with the Wöhler curves enable us an approximate determination of load limits responsible for a probable further crack propagation. By means of the proposed approach one can predict a damage probability of the faulty rotor-shaft system of arbitrary structure operating under various dynamic and quasi-static loads affecting a crack of various sizes and shaft locations. From the investigations performed for various crack axial locations on the shaft with respect to bearing supports of the single- and double-span rotor-shaft systems, it follows that a strength of the cracked zone is much more sensitive to normal stresses due to bending and axial oscillations than to tangential stresses caused by torsional vibrations.

KEY WORDS
Cracked rotor dynamics, fatigue life estimation, one- and three-dimensional model, coupled vibration analysis, stress distribution

1 INTRODUCTION

An investigation of crack development from the viewpoint of fatigue life estimation of the rotor is a very important task for reliable and safe exploitation of modern heavily affected rotating machines. Thus, recent fundamentals of machinery dynamics, fracture mechanics and material fatigue have to be employed to build proper mechanical models necessary for thorough studies of these phenomena. Nevertheless, except [1,2], it is rather difficult to find in the literature results of such investigations devoted to typical rotor-dynamics applications. Vibration analyses of cracked rotors are mostly focused on dynamic diagnostics problems in order to detect, localize and identify the size of a fault, which follows e.g. from [3-8]. For this purpose one-dimensional mechanical models of the rotor-shaft system are commonly applied, where the beam finite element approach seems to be the most reliable now. The three-dimensional finite element model has been applied in [6] for the cracked zone of the rotor-shaft, regarded there as a separated part from the entire rotating system, in order to determine the additional flexibility components of the weakened shaft caused by the transverse crack. In [1], by means of the finite element method, for bending vibrations only, there were determined the flexibility components of the rotor-shaft cracked zone as well as the stress intensity factors necessary for a fatigue life estimation. In [2] the investigations were focused on an optimization aspect from the viewpoint of minimization of fatigue stresses caused by stress concentration due to geometrical shape irregularities of the rotor-shafts.

The main target of the presented paper is a fatigue analysis and life estimation of cracked rotor-shafts carried out also by means of the one- and three-dimensional dynamical models. For this purpose static and dynamic loadings of the shaft cracked zone have to be determined first by means of numerical simulation of vibratory
behaviour of the rotating parts of the machine. Such a target can be credibly and effectively achieved by an application of the one-dimensional model of the entire rotor-shaft system, by use of which, similar to [6,7], the lateral-torsional-axial vibrations, coupled by the transverse crack, can be investigated. The three-dimensional finite element approach will be applied here also for the rotor-shaft cracked zone, but in order to determine maximum stress concentration and the stress intensity factors in the vicinity of the crack tip. Those quantities responsible for eventual initiation of further crack propagation are going to be determined as functions of current normal and tangential stresses in the cracked shaft segment resulting from the dynamic and quasi-static analysis of the entire rotor-shaft system carried out using the one-dimensional mechanical model. Such considerations are going to be performed for various rotor-shaft systems and for several axial crack locations on the shaft with respect to bearing supports as well as for several circumferential positions.

2 ASSUMPTIONS AND FORMULATION OF THE PROBLEM

In this paper, in order to obtain reliable results together with a high computational efficiency, dynamic and quasi static investigations of the entire rotating system are performed by means of the one-dimensional hybrid structural model consisting of continuous visco-elastic macro-elements and of discrete oscillators. Such a hybrid model of the double-span rotor-shaft system supported on three journal bearings, used here as an example, is presented in Fig. 1. By the use of this model eigenvalue analyses and numerical simulations of coupled lateral-torsional-axial vibrations of the cracked rotor-shaft are going to be carried out. Similarly as in [9], in this model successive cylindrical segments of the stepped rotor-shaft are substituted by means of flexurally, axially and torsionally deformable cylindrical macro-elements of continuously distributed inertial-visco-elastic properties. With a reasonable accuracy for practical purposes, the heavy rotors are represented by rigid rings attached to the shafts by means of mass-less membranes enabling rotations of these rings around their diameters as well as their translational displacements in the shaft axial direction, as shown in Fig. 1. Each journal bearing is represented by means of a dynamic oscillator of two degrees of freedom, where beyond the oil-film interaction also the visco-elastic properties of the bearing housing and foundation are taken into consideration. This bearing model makes possible to represent with relatively high accuracy kinetostatic and dynamic anisotropic and anti-symmetric properties of the oil-film in the form of constant stiffness and damping coefficients, [7]. In this system in the selected rotor-shaft segment there is considered a transverse crack implemented in the model by proper elastic connection of the respective adjacent left- and right-hand side part of the faulty shaft segment. Additional flexural, torsional and axial flexibilities introduced by the crack into the shaft are represented here by means of mass-less springs connecting the adjacent beam macro-elements substituting this cracked shaft segment. Stiffness values of these springs are determined according to [3] using the fundamentals of fracture mechanics. Such a crack model makes possible to take into consideration coupling effects between the torsional and axial vibrations of the rotor-shaft with its bending vibrations induced by residual unbalances of the rotating system.

Figure 1: Hybrid one-dimensional mechanical model of the three-bearing cracked rotor-shaft system.

In the hybrid model motion of cross-sections of each visco-elastic macro-element of the length \( l_i \) is governed by means of the partial differential equations derived using the Timoshenko and Rayleigh rotating beam theory for flexural motion and by the hyperbolic equations of the wave type, separately for torsional and axial motion. Similarly as in [9], mutual connections of the successive macro-elements creating the stepped shaft as well as their interactions with the supports and rigid rings representing the heavy rotors are described by equations of boundary conditions. These equations contain geometrical conditions of conformity for translational and rotational displacements of extreme cross-sections \( x=L_x=l_1+l_2+...+l_{i-1} \) of the adjacent \((i-1)\)-th and the \(i\)-th elastic macro-elements:
Then, for the “open” crack in the shaft cross-section \( E \) where the mentioned adjacent (equations of equilibrium for external forces and torques, static and dynamic unbalance forces and moments, lateral displacement in the horizontal direction, \( i = 1, 2, ..., n \), \( j \) is the imaginary number and \( n \) denotes the total number of macro-elements in the hybrid model.

The second group of boundary conditions are dynamic ones, which contain linear, non-linear and parametric equations of equilibrium for external forces and torques, static and dynamic unbalance forces and moments, inertial, elastic and external damping forces, support reactions and gyroscopic moments. For example, the dynamic boundary conditions formulated for the rotating Rayleigh beam, and describing a simple connection of the mentioned adjacent (1-1)-th and the i-th elastic macro-elements, have the following form:

\[
EI \frac{\partial^3 v_{i-1}(x,t)}{\partial x^3} - EI \frac{\partial^3 v_i(x,t)}{\partial x^3} = 0, \\
EI \frac{\partial^2 v_{i-1}(x,t)}{\partial x^2} - EI \frac{\partial^2 v_i(x,t)}{\partial x^2} = 0, \\
2GI \frac{\partial \theta_i(x,t)}{\partial x} - 2GI \frac{\partial \theta_{i-1}(x,t)}{\partial x} = 0, \\
EA \frac{\partial x_i(x,t)}{\partial x} - EA \frac{\partial x_{i-1}(x,t)}{\partial x} = 0,
\]

where \( I \) is the diametral geometric moment of inertia of the i-th macro-element cross-section of the area \( A \), and \( E, G \) denote Young and Kirchhoff moduli of the shaft material of the density \( \rho \).

The transverse crack in the given \( k \)-th shaft segment is also described by boundary conditions formulated for the extreme cross-sections of adjacent macro-elements representing this shaft segment. Then, their dynamic equations of equilibrium of forces and moments (2) remain valid. However, the geometric equations of displacement conformity (1) must be appropriately substituted by proper modified relations. If in the shaft the transverse crack is assumed, its “breathing” process usually has to be taken into consideration as e.g. in [7,8]. Then, for the “open” crack in the shaft cross-section \( x=x_L, x=x_L+\alpha \), the respective boundary conditions have the following form:

\[
EI \frac{\partial^3 v_{k-1}(x,t)}{\partial x^3} = \left[ k_{11}^* (t) + jk_{13}^* (t) \right] \Re[v_k - v_{k-1}] + j\left[ k_{33}^* (t) - jk_{13}^* (t) \right] \Im[v_k - v_{k-1}] + \left[ k_{16}^* (t) + jk_{36}^* (t) \right] \Re[\theta_k - \theta_{k-1}],
\]

\[
EI \frac{\partial^2 v_{k-1}(x,t)}{\partial x^2} = \left[ k_{22}^* (t) + jk_{24}^* (t) \right] \Re[\frac{\partial v_k}{\partial x} - \frac{\partial v_{k-1}}{\partial x}] + j\left[ k_{44}^* (t) - jk_{24}^* (t) \right] \Im\left[ \frac{\partial v_k}{\partial x} - \frac{\partial v_{k-1}}{\partial x} \right] + \left[ k_{25}^* (t) + jk_{45}^* (t) \right] \Re[\frac{\partial \theta_k}{\partial x} - \frac{\partial \theta_{k-1}}{\partial x}],
\]

\[
2GI \frac{\partial \theta_{k-1}}{\partial x} = c_{56} \left[ \theta_k - \theta_{k-1} \right] + \left[ k_{16}^* (t) \right] \Re[\frac{\partial v_k}{\partial x} - \frac{\partial v_{k-1}}{\partial x}] + \left[ k_{36}^* (t) \right] \Im[\frac{\partial v_k}{\partial x} - \frac{\partial v_{k-1}}{\partial x}],
\]

\[
EA \frac{\partial z_{k-1}}{\partial x} = c_{55} \left[ z_k - z_{k-1} \right] + \left[ k_{25}^* (t) \right] \Re[\frac{\partial v_k}{\partial x} - \frac{\partial v_{k-1}}{\partial x}] + \left[ k_{45}^* (t) \right] \Im[\frac{\partial v_k}{\partial x} - \frac{\partial v_{k-1}}{\partial x}],
\]

where: \( k_{11}^* (t) = \frac{1}{2} \left( c_{11} + c_{33} \right) + \frac{1}{2} \left( c_{11} - c_{33} \right) \cos(2\alpha) - c_{13} \sin(2\alpha) \),

\( k_{13}^* (t) = \frac{1}{2} \left( c_{11} - c_{33} \right) \sin(2\alpha) + c_{13} \cos(2\alpha) \), \( k_{16}^* (t) = c_{16} \cos(\alpha) - c_{36} \sin(\alpha) \),

\( k_{33}^* (t) = \frac{1}{2} \left( c_{11} + c_{33} \right) - \frac{1}{2} \left( c_{11} - c_{33} \right) \cos(2\alpha) + c_{13} \sin(2\alpha) \), \( k_{36}^* (t) = c_{16} \sin(\alpha) + c_{36} \cos(\alpha) \),

\( k_{22}^* (t) = \frac{1}{2} \left( c_{22} + c_{44} \right) + \frac{1}{2} \left( c_{22} - c_{44} \right) \cos(2\alpha) - c_{24} \sin(2\alpha) \),

\( k_{24}^* (t) = \frac{1}{2} \left( c_{22} - c_{44} \right) \sin(2\alpha) + c_{24} \cos(2\alpha) \), \( k_{25}^* (t) = c_{25} \cos(\alpha) - c_{45} \sin(\alpha) \),

\( k_{44}^* (t) = \frac{1}{2} \left( c_{22} + c_{44} \right) - \frac{1}{2} \left( c_{22} - c_{44} \right) \cos(2\alpha) + c_{24} \sin(2\alpha) \), \( k_{45}^* (t) = c_{25} \sin(\alpha) + c_{45} \cos(\alpha) \),

\( \alpha = \Omega t + \alpha_p \), and \( \alpha_p \) denotes the angle of crack circumferential position on the shaft.
In the above equations the constant stiffness coefficients $C_{11}$, $C_{13}$, $C_{33}$, $C_{16}$, $C_{36}$, $C_{22}$, $C_{24}$, $C_{44}$, $C_{25}$, $C_{45}$, $C_{55}$, $C_{66}$ are obtained according to [3,7] using the Paris equation and by means of inversion of the symmetrical crack local flexibility matrix determined by partial differentiation of the strain energy density function with respect to loading forces and moments. In this paper the so called “hinge” crack model is proposed, which is admissible for relatively small cracks in the shaft. Then, the open/closed-criterion of crack “breathing” can be described by the following relation formulated in the co-ordinate system rotating with the shaft:

$$\Delta \varphi_k (L_{cr}, t) > 0, \text{ i.e. the crack is open and } \Delta \varphi_k (L_{cr}, t) \leq 0, \text{ i.e. the crack is closed},$$  \(4\)

where: $\Delta \varphi_k (L_{cr}, t) = \left( \frac{\partial u_k(x,t)}{\partial x} - \frac{\partial u_{k-1}(x,t)}{\partial x} \right) \cos(\omega t) + \left( \frac{\partial w_k(x,t)}{\partial x} - \frac{\partial w_{k-1}(x,t)}{\partial x} \right) \sin(\omega t)$ for $x = L_{cr}$.

It is assumed that for the “closed” crack, i.e. for (4), the geometric boundary conditions (1) are valid instead of relations (3). Moreover, if the crack is closed and the shaft cracked vicinity is compressed due to the axial vibrations, the boundary condition (1), is valid. However, for the vicinity of the cracked shaft in tension this relation must be substituted by (3). It is to remark that here, the problem of cracked shaft is formulated as a parametric-nonlinear one, because the considered system changes its configuration depending on, whether the crack is temporarily “open” or “closed”, [7,8]. The above equations of the boundary conditions (2), (3) are formulated using the Rayleigh beam theory. In an analogous way there are derived such boundary conditions by means of the Timoshenko beam theory.

Equations (3) demonstrate, how the crack in the shaft couples the rotor-shaft bending vibrations with the torsional and axial vibrations. In order to perform an analysis of natural elastic vibrations, all the forcing, viscous, parametric and unbalance terms standing in the boundary conditions have been omitted. Due to truncation of these terms, it follows from (3) that the bending, torsional and axial vibrations of the rotor shaft system are mutually uncoupled. According to the above, the elastic bending, torsional and axial eigenvalue problems can be solved separately. Thus, one obtains separate characteristic equations for the considered three eigenvalue problems. These are

$$A(\omega) \mathbf{B} = \mathbf{0} \text{ for the rotor-shaft bending vibrations coupled with bending vibrations of the heavy rotors elastically attached to the shaft by the membranes,}$$

$$C(\omega) \mathbf{D} = \mathbf{0} \text{ for the rotor-shaft torsional vibrations,}$$

$$E(\omega) \mathbf{F} = \mathbf{0} \text{ for the rotor-shaft axial vibrations coupled with translational vibrations of the heavy rotors elastically attached to the shaft by the membranes,}$$

where $A(\omega)$, $C(\omega)$ and $E(\omega)$ are the complex and real characteristic matrices, respectively, and $\mathbf{B}$, $\mathbf{D}$, $\mathbf{F}$ denote the vectors of unknown constant coefficients in the analytical eigenfunctions defined in [3,9]. Thus, the determination of natural frequencies reduces to the search for values of $\omega$, for which the characteristic determinants of matrices $A$, $C$ and $E$ are equal to zero. The bending, torsional and axial eigenmode functions are then obtained by solving respective equations (5).

The solution for the forced vibration analysis has been obtained using the analytical-computational approach demonstrated in detail in [9]. Solving the differential eigenvalue problem for the linearized orthogonal system and an application of the Fourier solutions in the form of series lead to the set of modal equations in the Lagrange co-ordinates

$$\mathbf{M}(\Omega t) \ddot{\mathbf{r}}(t) + \mathbf{C}(\Omega t, \Omega t) \dot{\mathbf{r}}(t) + \mathbf{K}(\partial \varphi(t), \Omega t) \mathbf{r}(t) = \mathbf{F}(t, \Omega 2, \Omega t),$$  \(6\)

where: $\mathbf{M}(\Omega t) = \mathbf{M}_0 + \mathbf{M}_\omega (\Omega t)$,

$$\mathbf{C}(\Omega t, \Omega t) = \mathbf{C}_0 + \mathbf{C}_\omega (\Omega t) + \mathbf{C}_\gamma (\Omega t)$,$$  \(6\)

$$\mathbf{K}(\partial \varphi(t), \Omega t) = \mathbf{K}_0 + \mathbf{K}_b + \mathbf{K}_{cr}(\Delta \varphi_k(L_{cr}, t), \Omega t).$$

The symbols $\mathbf{M}_0$, $\mathbf{K}_0$ denote, respectively, the constant diagonal modal mass and stiffness matrices, $\mathbf{C}_0$ is the constant symmetrical damping matrix and $\mathbf{C}_\omega (\Omega t)$ denotes the skew-symmetrical matrix of gyroscopic effects. The terms of the unbalance effects are contained in the symmetrical matrix $\mathbf{M}_\omega (\Omega t)$ and in the non-symmetrical matrix $\mathbf{C}_\omega (\Omega t)$. Anti-symmetric elastic properties of the journal bearings are described by the skew-symmetrical matrix $\mathbf{K}_\omega$, non-linear and parametric properties of the breathing crack are described by the symmetrical matrix $\mathbf{K}_{cr}(\Delta \varphi_k(L_{cr}, t), \Omega t)$ of periodically variable coefficients and the symbol $\mathbf{F}(t, \Omega^2, \Omega t)$ denotes the external excitation vector, e.g. due to the unbalance and gravitational forces. The Lagrange co-ordinate vector $\mathbf{r}(t)$ consists of subvectors of the unknown time functions in the Fourier solutions. In order to obtain the system's dynamic response Eqs. (6) are solved by means of a direct integration. The number of Eqs. (6) corresponds to the number of eigenmodes taken into consideration, because the forced bending, torsional and axial vibrations of the rotor shaft are mutually coupled and thus, the total number of Eqs. (6) to solve is a sum of all bending, torsional and axial eigenmodes of the rotor shaft model in the range of frequency of interest. These equations are mutually coupled by the parametric and anti-symmetrical terms regarded as external excitations expanded in series in the base of orthogonal analytical eigenfunctions. A fast convergence of the applied Fourier solutions enables us to
reduce the appropriate number of the modal equations to solve, in order to obtain a sufficient accuracy of results in the given range of frequency. Such a mathematical description of the investigated hybrid model is formally strict, demonstrates clearly the qualitative system properties and is much more convenient for a stable and efficient numerical simulation than the classical one-dimensional beam finite element formulation.

By means of such one-dimensional structural model of the entire rotor-shaft system, the lateral-torsional-axial vibrations, coupled by the crack, can be investigated in order to determine shaft dynamic and static loadings necessary for fatigue analysis of the shaft cracked zone. This zone must be modelled very exactly taking into consideration the crack shape and stress distribution in the vicinity of the crack. Thus, the three-dimensional finite element method (FEM) has been applied with appropriate adaptation of mesh density. The geometry and dimensions of the analyzed faulty shaft segment are presented in Figure 2a. The considered part of the cracked shaft was discretized by 12212 tetrahedral 4-node finite elements, which is shown in Figure 2b.

**Figure 2:** The three-dimensional finite element model of the rotor-shaft cracked zone: geometry and dimensions (in mm) of the shaft segment (a), the discretized part of the shaft (b).

The bending, torsional and axial loads in the form of time histories or amplitudes of dynamic forces and moments, obtained by means the one-dimensional hybrid model, are then imposed to the three-dimensional FEM model to act here as quasi-static excitations. Thus, the left-hand edge of the shaft segment shown in Fig. 2 has been assumed clamped and the right-hand edge is loaded by the bending moments acting in the $0xy$ and $0xz$ planes, the torsional moment acting around the $0x$ axis, as well as by the axial force acting along the $0x$ axis. Such an approach enables us to determine the time- and space-varying stress distribution in the cracked shaft segment. This stress distribution is regarded here as an input data for fatigue life estimation for the considered rotor-shaft as well as for prediction of the crack propagation process. This task is solved using the classical Wöhler curves incorporated into the finite element code FEAP developed by [10] and supplemented by the Authors of [11]. The parameters of the Wöhler curve for the shaft material, i.e. for an alloy steel of Young’s modulus $E=2.1 \times 10^{11}$ Pa and Poisson’s ratio $\nu=0.3$, are following: the saturation flow stress corresponding to the material ultimate stress defining its static strength $S_u=520 \times 10^6$ Pa and the threshold stress is equal to $S_t=260 \times 10^6$ Pa, [11]. In order to evaluate a probability of further crack propagation due to dynamic loads transmitted by the defected shaft segment, the stress intensity factors at the crack tip are going to be numerically determined. For this purpose also the singular finite elements have been introduced to model the crack tip vicinity. Then, according to [12], the stress intensity factor $K_I$ is obtained from the following relation

$$K_I = \sqrt{\frac{E}{2(1-\nu^2)}} \frac{q^T R \partial l}{l} q,$$

where $q$ denotes the displacement vector of the three-dimensional finite element model shown in Fig. 2b, $R$ is the stiffness matrix and $l$ denotes the perturbation of the node location on the crack tip axis.

### 3 NUMERICAL RESULTS

In the first step numerical calculations have been performed for a single-span rotor-shaft system with two heavy identical rotors of mass 200 kg each located on the shaft between two journal bearings, as shown in Fig. 3. The stepped-rotor shaft of total length 2.315 m is modelled by means of $n=14$ continuous macro-elements. The bearing span is equal to 1.6 m and the total mass of the considered rotor-shaft system amounts ca. 590 kg. In this system the transverse crack of depth ratio $a/D=0.3$ is assumed in the shaft cross-section corresponding to the middle of the bearing span, as shown in Fig. 3, where $D=0.12$ m.

The system dynamic response has been obtained for the range of system exploitation rotational speeds 500÷12000 rpm for four circumferential crack positions on the shaft: $\alpha_c=0$, 90, 180 and 270 deg. The only assumed source of dynamic external excitation are static residual unbalances of the heavy rotors equal to 2.772 gm each and mutually shifted by the phase angle $\Delta=180$ deg. Thus, the torsional and axial vibrations can be regarded here as an output effect caused by bending vibrations of the rotor-shaft. The main target of the carried out numerical simulations is an investigation of the most severe dynamic and quasi-static load of the cracked...
zone in order to predict its fatigue life. The quantities of particular interest in these investigations are the lateral vibration displacements of the shaft at the heavy rotor locations, bending and torsional moments as well as the longitudinal force transmitted by the shaft cracked zone. The selected results of simulations are presented in Fig. 4 in the form of amplitude characteristics of the lateral displacements and of the dynamic torque expressed as functions of the constant rotational speed values. The amplitudes of all mentioned quantities are regarded as maximum fluctuation values with respect to their average values of the steady-state dynamic response. In Fig. 4a there are shown the lateral displacement amplitude characteristics for the 1st and 2nd heavy rotor. Both plots are almost identical and they are characterized by one resonance peak at the rotational speed ca. 9500 rpm approximately corresponding to the system 6th bending natural frequency 166.4 Hz determined by means of the
Rayleigh- and confirmed by the Timoshenko beam theory. The eigenform of the considered two-bearing rotor-shaft system, corresponding to this frequency, is presented in Fig. 5b, where by the black line the vertical projection of this eigenfunction is depicted and by the grey line – its horizontal projection. It is to remark that the 6\textsuperscript{th} eigenform with a node close to the crack is not sensitive to parametric effects caused by the crack, contrary to the fundamental bending eigenform with frequency 38.599 Hz, shown in Fig. 5a, weakly induced below 3500 rpm. Moreover, the respective plots in Fig. 4a obtained for four considered crack position angles $\alpha$, almost overlay, which means that in the studied overweight system the circumferential crack locations do not influence the rotor-shaft bending vibrations. The amplitude characteristics of the dynamic torque transmitted by the cracked shaft zone, presented in Fig. 4b, are characterized by two resonance peaks at the rotational speed ca. 8000 rpm corresponding to the system 1\textsuperscript{st} torsional natural frequency 134.7 Hz and at 4000 rpm corresponding to one half of this frequency. The respective first torsional eigenform is shown in Fig. 5c. Here, these greater torsional vibration amplitudes at the mentioned rotational speed zones are excited by parametric resonances of the 1\textsuperscript{st} and the 0.5\textsuperscript{th} order, respectively, induced by coupling effects caused by the crack.

### Table 1: Coupled vibration amplitudes of the two-bearing cracked rotor-shaft system.

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<td>Steady-state vibration amplitudes of:</td>
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<td></td>
<td>Parametric resonance of the 0.5\textsuperscript{th} order</td>
<td>Out of resonance operation</td>
<td>Parametric resonance of the 1\textsuperscript{st} order</td>
<td>Bending ordinary resonance</td>
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<tr>
<td>bending moment [Nm] – in vertical plane to the crack tip</td>
<td>1000</td>
<td>1100</td>
<td>1000</td>
<td>1050</td>
</tr>
<tr>
<td>bending moment [Nm] – in horizontal plane to the crack tip</td>
<td>1000</td>
<td>1100</td>
<td>1100</td>
<td>1050</td>
</tr>
<tr>
<td>torsional moment [Nm]</td>
<td>26</td>
<td>6</td>
<td>130</td>
<td>27</td>
</tr>
<tr>
<td>axial force [N]</td>
<td>690</td>
<td>720</td>
<td>720</td>
<td>700</td>
</tr>
<tr>
<td>number of cycles till damage</td>
<td>1172</td>
<td>56</td>
<td>478</td>
<td>419</td>
</tr>
<tr>
<td>SIF $K_1$ [MPa$\cdot$m$^{1/2}$]</td>
<td>160.4</td>
<td>176.5</td>
<td>160.4</td>
<td>168.1</td>
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From the obtained results of simulation it follows that in the whole range of rotational speeds amplitudes of the bending moment observed in the coordinate system rotating with the shaft are almost constant and amount ca. 1000÷1100 Nm, both in the vertical and horizontal plane to the crack tip. Also amplitudes of the axial force remain unchanged and amount ca. 700 N for all rotational speeds within the considered 300÷12000 rpm. Only amplitudes of the torsional response are sensitive to resonances, in contrast to the amplitudes of bending moments and axial force. This fact can be explained by the essential kinetostatic deflection of the considered typical overweight single-span rotor-shaft system with two heavy rotor-disks attached to the shaft between the bearings, Fig. 3. Here, in the vicinity of the crack in the middle of the bearing span rotation of the significantly deflected shaft causes much greater synchronous fluctuation of the bending moment than the shaft bending vibrations due to the residual unbalances. This predominant oscillation of the bending moment of almost constant amplitude, according to Eqs. (3)\textsuperscript{2} and (3)\textsuperscript{4}, influences the axial response which is also characterized by force amplitude values very weakly dependent on the system rotational speed. Table 1 contains amplitude values of the steady-state coupled bending-torsional-axial vibrations obtained for four noteworthy rotational speeds of the considered two-bearing cracked rotor-shaft system, i.e. for the three abovementioned resonant speeds as well as for the non-resonant operating conditions at 6000 rpm. From these numerical values it follows that, in the case of the overweight single-span rotor-shaft system considered, the normal stresses due to the bending moments and the axial force are predominant loads affecting the shaft cracked zone. Then, the proper fatigue life prediction...
has been performed for such mutual relations of the response amplitudes assuming the most inconvenient conditions, where the fluctuations of bending, torsional and axial loads reach their extremes simultaneously. Table 1 contains the results of material fatigue analysis performed by means of the three-dimensional FEM model of the shaft cracked zone. These are the limit numbers of oscillation cycles before fatigue damage obtained using the Wöhler curves as well as the stress intensity factors $K_I$. All listed numbers of cycles are very small, which means that the synchronously vibrating and significantly deflected overweight shaft is close to fatigue damage. This fact is confirmed by all SIF values in Table 1 essentially greater than 80 MPa\(\cdot\)m\(^{1/2}\), i.e. exceeding the fracture toughness limits for high-strength alloy steels applied for well-designed rotor-shafts.

In order to perform a fatigue life prediction for the same cracked shaft zone but operating under different mutual proportions of the response amplitudes, the structure of the considered rotor-shaft system has been modified by the introduction of a third bearing in the mid-span and by keeping the crack in the vicinity of this additional support, as shown in Fig. 1. The visco-elastic parameters of this third bearing are assumed the same as for the two remaining bearings. For this purpose the shaft had to be slightly modified in order to use the same journal diameter as in the case of both remaining bearings as well as it being necessary to introduce a rigid coupling to keep the considered system realistic. Thus, the entire rotor-shaft of length 2.315 m (same as before), is represented by $n=19$ visco-elastic continuous macro-elements. All remaining parameters are assumed identical to the previous example. Thus, the double-span rotor-shaft system depicted in Fig 1 is obtained, supported on three journal bearings with the same two heavy rotors, i.e. each in its “new” bearing span equal to 0.8 m. Here, the total mass of the modified rotor-shaft system is also close to ca. 590 kg.

The modifications listed above have almost no influence on the torsional eigenvibration properties and very little influence on the axial eigenvibration properties. For example, the first torsional eigenform, presented in Fig. 6c, is characterized by a similar natural frequency to that in the previous case. Also, all bending non-symmetrical eigenfunctions of the two-bearing rotor-shaft system, i.e. with nodes in the middle of the bearing span, remained almost unchanged with respectively very similar natural frequency values, which confirms the 6th lateral eigenform depicted in Fig. 6b. However, the additional third bearing #2, Fig. 1, has “stiffened” the symmetrical eigenfunctions by an increase of corresponding natural frequencies. For example, the eigenfrequency of the 1st eigenform, presented in Fig. 6a, increased to 74.101 Hz for the three-bearing system, assumed in this way, in comparison with 38.199 Hz for the two-bearing one, Fig. 5a.

![Figure 6: The most intensively excited eigenforms of the three-bearing rotor-shaft system.](image)

According to the above, for the assumed in this way three-bearing rotor-shaft system and for the identical heavy rotor static unbalances mutually shifted by the phase angle $\Delta=180$ deg, by means of numerical simulations of the bending-torsional-axial vibrations coupled by the transverse crack of the ratio $a/D=0.3$ for $D=0.12$ m in the rotational speed range 500÷12000 rpm similar amplitude characteristics of the shaft lateral displacements and dynamic torque have been obtained as these presented in Figs 4a and 4b. Then, for the analogous (as in the previous case) four noteworthy rotating speeds, Table 2 contains amplitude values of the steady-state coupled dynamic response. From these numerical values it follows that in the case of the double-span rotor-shaft system, the amplitudes of the bending moments and of the axial force affecting the shaft cracked zone are significantly dependent on the shaft rotational speed and much smaller than these contained in Table 1. Moreover, under parametric resonance, i.e. for $N=3972$ and 7944 rpm, they are comparable with the amplitude values of the torsional moment. However, for the two remaining rotational speeds the influence of the torsional load is negligible, similarly to the previous case of the single-span rotor-shaft system. These smaller bending, axial and torsional loads of the cracked shaft segment result in at least few hours-long or even almost unlimited fatigue life for $N=3972$, 6000 and 7944 rpm, Table 2. Only in the case of the ordinary bending resonance, due to the relatively greater amplitudes of the bending moments, and the axial force, the limit number of cycles till damage is equal to $\sim31000$, which for $N=9990$ rpm predicts only ca. 3 minute long operation before fatigue damage. This is confirmed by the corresponding SIF which exceeds 80 MPa\(\cdot\)m\(^{1/2}\), where all remaining SIF values contained in Table 2 are lower than the abovementioned fracture toughness limits for high-strength steels.
The above theoretical comparison of dynamic and fatigue analyses carried out for the two considered rotor-shaft systems confirms the fact observed in practice that different structures of vibrating objects together with different crack locations with respect to the bearing supports introduce other operational conditions and damage probability of the defected shaft. Other operational conditions and damage probability can be obtained for different external excitations due to the mutual phase shift $\Delta$ of the rotor-disk unbalances while their magnitudes remain unchanged. In the next example, the double-span three-bearing rotor-shaft system presented in Fig. 1 will be excited by the residual unbalances of the same abovementioned numerical values, but mutually oriented “in phase”, i.e. for the phase shift angle $\Delta=0$ deg. Then, the fundamental first bending eigenform of natural frequency 74.101 Hz, depicted in Fig. 6a, is more significantly induced, instead of the 6th non-symmetrical eigenforms plotted in Figs. 5b and 6b. In this case, beyond the abovementioned four noteworthy rotational speeds, Table 3 also contains the amplitudes of the steady-state dynamic response obtained for the ordinary resonance of bending vibrations at the system 1st critical speed equal to $N=4470$ rpm. It should be emphasized that all amplitude values for the selected five most important rotational speeds are respectively much smaller than those contained in Tables 1 and 2. Moreover, the amplitudes of torsional response in Table 3 obtained for both parametric resonance conditions, i.e. for $N=3972$ and 7944 rpm, are significantly greater than the corresponding amplitudes of the bending and axial vibrations. Nevertheless, in all five cases of the rotational speed there is predicted many hours-long or almost infinite exploitation time of the rotor-shaft system. Here, the predominant tangential stresses in the cracked zone seem to have no essential influence on its fatigue life, which follows from the infinite numbers of cycles till damage, included in Table 3. This fact has been also confirmed by the SIF values, contained in Table 3, where they are all essentially smaller than the fracture toughness limits.

### Table 2: Vibration amplitudes of the three-bearing cracked rotor-shaft system (unbalances “in anti-phase”).

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Steady-state vibration amplitudes of:</td>
<td>Parametric resonance of the 0.5th order</td>
<td>Out of resonance operation</td>
<td>Parametric resonance of the 1st order</td>
<td>Bending ordinary resonance</td>
</tr>
<tr>
<td>bending moment [Nm] – in vertical plane to the crack tip</td>
<td>100</td>
<td>220</td>
<td>340</td>
<td>740</td>
</tr>
<tr>
<td>bending moment [Nm] – in horizontal plane to the crack tip</td>
<td>95</td>
<td>210</td>
<td>370</td>
<td>730</td>
</tr>
<tr>
<td>torsional moment [Nm]</td>
<td>100</td>
<td>13</td>
<td>200</td>
<td>11.5</td>
</tr>
<tr>
<td>axial force [N]</td>
<td>50</td>
<td>145</td>
<td>205</td>
<td>590</td>
</tr>
<tr>
<td>number of cycles till damage</td>
<td>$&gt; 2\times 10^6$</td>
<td>$&gt; 2\times 10^6$</td>
<td>$&gt; 2\times 10^6$</td>
<td>30999</td>
</tr>
<tr>
<td>SIF $K_i$ [MPa-m^1/2]</td>
<td>16.1</td>
<td>35.3</td>
<td>55.6</td>
<td>118.2</td>
</tr>
</tbody>
</table>

4 FINAL REMARKS AND CONCLUSION

In the paper a fatigue life prediction has been performed for the cracked zone of single- and double-span rotor-shaft systems, under coupled bending-torsional-axial vibrations. The considerations were carried out for transverse cracks of various circumferential positions on the shaft as well as for various axial locations with respect to bearing supports. For dynamic analyses of the entire vibrating system, the hybrid one-dimensional mechanical model, consisting of structural continuous visco-elastic macro-elements and discrete oscillators, was...
applied. In order to represent the crack geometry and to determine the stress distribution of the shaft cracked zone with sufficient accuracy, the faulty shaft segment has been modelled by means of three-dimensional finite elements. For the material fatigue life estimation two approaches were used. In the first case, maximal normal and tangential stresses concentrated at the crack tip were determined in order to refer them to the Wöhler curves. In the second case, the crack tip was modelled by means of singular finite elements which enabled us to calculate stress intensity factors regarded here as criteria of possible further propagation of the crack. For both listed cases, as input data for the fatigue life prediction, the amplitudes of the bending moments, axial force and torsional moment transmitted by the shaft cracked zone, and obtained by means of numerical simulations using the one-dimensional hybrid model, were applied.

From the computation results obtained for various resonant and non-resonant operating conditions of the considered rotor-machines it follows that normal stresses in the shaft cracked zone due to bending and axial dynamic loads play a predominant role for material fatigue life, whereas tangential stresses caused by torsional vibrations are of secondary importance. Moreover, the performed theoretical investigation has confirmed the fact observed in practice that a transverse crack can exist in the steel shaft in a one position with respect to the bearing support or propagate very quickly in another position. It can be explained by some strength reserve of the un-cracked part of a cross-section of usually oversized rotor-shafts, where actual magnitude of stress at the crack tip depends on the particular investigated case. In the example demonstrated in the paper the medium-size crack located in the bearing mid-span of the strongly overweight shaft can break down immediately or remain unchanged for a long time when located close to the support in the multi-bearing rotor-shaft system.

On the one hand, the proposed method is computationally efficient and seems to be credible for engineering applications, since it is based on the well assumed theoretical one- and three dimensional mechanical models cooperating with the advanced material fatigue life estimation routines. But on the other hand, the fatigue analysis carried out has rather a quasi-static character, because it uses only amplitude values of the dynamic steady-state response instead of its fluctuating time histories. However, it is worth noting the good agreement between respective results of the fatigue life prediction obtained by the use of the Wöhler curves and of the stress intensity factors for all the considered computational examples, one should be conscious that determination of the maximal stresses and stress intensity factors at the crack tip is quite approximate since the results are influenced by the discretization mesh densities. Nevertheless, this deficiency is still acceptable for engineering applications because of the lack of better approaches in this field, which should be developed by means of more advanced methods of modelling of the cracked structures used on the fundamentals of physics of metals.

REFERENCES