International Journal of Theoretical Physics, Vol. 42, No. 4, April 2003 (© 2003)

Self-Balance Equations and Bianchi-Type Distortions in the Theory of Dislocations

A. Trzęsowski[1]

Received December 19, 2002

We deals with two aspects of the geometric description of continuized Bravais monocrystals with many dislocations. First, a triad of vector fields is distinguished constituting a basis for the C^{∞}-module of smooth vector fields tangent to the body. This moving frame defines its object of anholonomity as well as an intrinsic ("material") Riemannian metric of the body. Second, these geometric objects are used to define both the notions of principal congruence of Volterra-type effective dislocations and principal local Burgers vector associated with these congruences. The main topics discussed are (i) self-balance equations of dislocations and secondary point defects generated by distributions of these dislocations; (ii) a linkage of the Bianchi classification of three-dimensional real Lie algebras with the physical classification of principal local Burgers vectors.

KEY WORDS: self-balance equations; Bianchi-type distortions; dislocations.

1. INTRODUCTION

It is known that the occurrence of many dislocations in a Bravais monocrystal generates a bend in originally straight lattice lines of this crystal (Orlov, 1983). The distorted lattice lines can be represented, in a continuous limit approximation, by a system of three independent congruences of curves, and tangents to these curves define *local crystallographic directions* of the continuized Bravais crystal with dislocations. Planes spanned by two local crystallographic directions are *local crystal planes.* If a *crystallographic congruence*, that is, a congruence of lattice lines, is normal (Eisenhart, 1964) and its curves are orthogonal to local crystal planes everywhere, the curves are orthogonal trajectories of a family of *crystal surfaces* of the continuously dislocated Bravais crystal (Trzęsowski, 2001).

More precisely, we are dealing with the following geometric description of these crystals (Trzęsowski, 2001). Let $\mathcal{B} \subset E^3$ denote a *body* identified with its distinguished spatial configurations being an open subset of the three-dimensional Euclidean point space E^3 and contractible to a point. We will consider curvilinear

[1] Institute of Fundamental Technological Research, Polish Academy of Sciences, Świętokrzyska 21, 00-049 Warsaw, Poland.

coordinate systems $X = (X^A; A = 1, 2, 3)$ on $\mathcal{B}$ such that $[X^A] = \mathrm{cm}$. Let σ_A denote a coordinate curve

$$\sigma_A : (-\varepsilon, \varepsilon) \ni s \to X^{-1}(X(p) + s \in_A) \in \mathcal{B},$$
$$\in_A = (\delta_{AB}; B \to 1, 2, 3), \quad \sigma_A(0) = p \in \mathcal{B}, [s] = \mathrm{cm}, \tag{1.1}$$

with the tangent vector $c_A(p)$ defined by

$$c_A(p) = \dot{\sigma}_A(0) \in E^3, \quad [c_A(p)] = \mathrm{cm}^{-1}, \tag{1.2}$$

where $\dot{\sigma}_A = d\sigma_A/ds$ and E^3 denotes a three-dimensional Euclidean vector space identified with the space of all translations in E^3. In differential geometry the vector field $c_A : \mathcal{B} \to E^3$ is usually identified with the partial derivative operator $\partial_A = \partial/\partial X^A$. In this sense, a moving frame $\Phi = (E_a; a = 1, 2, 3)$ on $\mathcal{B}$ can be identified with the following first-order differential operators:

$$E_a = \underset{a}{e}{}^A \partial_A, \quad \underset{a}{e}{}^A \in C^\infty(\mathcal{B}), \quad [E_a] = \mathrm{cm}^{-1}. \tag{1.3}$$

The moving frame Φ can be considered as the one defining a system of three independent congruences of *lattice lines* of the continuous Bravais crystal with dislocations and Φ is called then a *Bravais moving frame*.

Next, we can define the nondimensional tensor measure $S[\Phi]$ of anholonomity of Φ such that $S[\Phi] = 0$, iff dislocations are absent. Namely, if $\Phi^* = (E^a; a = 1, 2, 3)$ is the Bravais moving coframe dual to Φ of Eq. (1.3)

$$E^a = \overset{a}{e}{}_A dX^A, \quad \overset{a}{e}{}_A \in C^\infty(\mathcal{B}),$$
$$\langle E^a, E_b \rangle = \overset{a}{e}{}_A \underset{b}{e}{}^A = \delta^a_b; \quad [E^a] = [dX^A] = \mathrm{cm}, \tag{1.4}$$

then the tensor field $S[\Phi]$ defined by

$$S[\Phi] = dE^a \otimes E_a = S^c_{ab} E^a \otimes E^b \otimes E_c$$
$$S_{ab}{}^c \in C^\infty(\mathcal{B}), \quad [S_{ab}{}^c] = \mathrm{cm}^{-1} \tag{1.5}$$

and the object of anholonomity $C_{ab}{}^c \in C^\infty(\mathcal{B})$ of Φ defined by

$$[E_a, E_b] = E_a \circ E_b - E_b \circ E_a = C^c_{ab} E_c \tag{1.6}$$

are related according to the following formula (Yano, 1958):

$$S_{ab}{}^c = -\frac{1}{2} C^c_{ab}. \tag{1.7}$$

It follows from (1.5)–(1.7) that

$$S[\Phi] = 0 \quad \text{iff} \quad E_a = \partial/\partial \xi^a, \quad a = 1, 2, 3, \tag{1.8}$$

where $\xi = (\xi^a)$ is a coordinate system of $\mathcal{B}$. Thus, the tensor field $S[\Phi]$ can be accepted as a measure of the distortion of lattice lines due to dislocations. The lack of this distortion is identified with the lack of dislocations.

The occurrence of dislocations is accompanied with the appearance of *secondary point defects* generated by the distribution of these dislocations (e.g., because of intersections of dislocation lines—Oding, 1961). Consequently, the metric properties of Bravais crystals with many dislocations are in general non-Euclidean. It can be modeled by the assumption that the considered body is additionally endowed with such Riemannian metric that reduces to a Euclidean metric when dislocations are absent. This *intrinsic metric tensor* can be taken in the following form (Trzęsowski, 1993, 2001):

$$g[\Phi] = \delta_{ab} E^a \otimes E^b = g_{AB} dX^A \otimes dX^B,$$
$$g_{AB} = \overset{a}{e}{}_A \overset{b}{e}{}_B \, \delta_{ab}, \quad [g_{AB}] = [1]. \tag{1.9}$$

The secondary point defects influence the slip phenomenon (Trzęsowski, 2001). It can be modeled by means of the treatment of lattice lines and crystal surfaces as those located in the Riemannian *material space* $\mathcal{B}_g = (\mathcal{B}, g[\Phi])$ associated with the considered distribution of dislocations (Trzęsowski, 2001).

It ought to be stressed that it is usually assumed that dislocations have no influence on metric properties of a crystal (Kröner, 1986). In the proposed approach, we assume that dislocations have no influence on local metric properties (of a crystal) only. Namely, the base vector fields of a Bravais moving frame are considered as those defining the local crystallographic directions as well as local scales of the Riemannian internal length measurement along these directions. It is a representation of the *short-range order* of continuously dislocated crystal with secondary point defect. The above-defined tensor measure of anholonomity and the curvature tensor (e.g., Eisenhart, 1964) of the Riemannian material space represent the *long-range distortion* of this crystal (Trzęsowski, 1994). The main topics of this paper, discussed in accordance with this model of the long-range distortion due to dislocations are (i) self-balance equations of dislocations and secondary point defects; (ii) a linkage of the Bianchi classification of three-dimensional real Lie algebras with the physical classification of congruences of dislocation lines associated with these algebras.

2. ANHOLONOMITY AND DENSITIES OF DISLOCATIONS

Let us consider the tensor measure of anholonomity $S[\Phi]$ of (1.5) as a tensor field on the Riemannian material space $\mathcal{B}_g$ (Section 1). We can introduce then the so-called *dislocation density tensor* α as an geometric object defined on this material space (Trzęsowski, 1984, 2001)

$$\boldsymbol{\alpha} = \alpha^{ab} \boldsymbol{E}_a \otimes \boldsymbol{E}_b, \quad [\boldsymbol{\alpha}] = \mathrm{cm}^{-3},$$
$$\alpha^{ab} = e^{acd} S_{cd}{}^{b}, \quad [\alpha^{ab}] = \mathrm{cm}^{-1}, \tag{2.1}$$

where $e^{abc} \overset{*}{=} \in^{abc}$ denotes the permutation symbol $\in^{abc}$ considered as a contravariant 3-vector density of weight $+1$ in $\mathcal{B}_g$ (Gołąb, 1966); $\overset{*}{=}$ means that a relation is valid in a distinguished coordinate system (or a base). The component α^{ab} of (2.1) can be written in the following form:

$$\begin{aligned} \alpha^{ab} &= \gamma^{ab} + \omega^{ab}, \\ \gamma^{ab} &= \alpha^{(ab)}, \quad \omega^{ab} = \alpha^{[ab]} = \frac{1}{2} t_c e^{cba}, \end{aligned} \tag{2.2}$$

and

$$t_a = e_{abc} \alpha^{bc} = C^c_{ac}, \tag{2.3}$$

where $e_{abc} \overset{*}{=} \in_{abc}$ denotes the permutation symbol $\in_{abc}$ $(=\in^{abc})$ considered as a covariant 3-vector density of weight—1 in $\mathcal{B}_g$ (Gołąb, 1966).

Thus, according to (1.5)–(1.7) and (2.1)–(2.2), the object of anholonomity C^c_{ab} can be written in terms of the dislocation density tensor

$$C^c_{ab} = t_{[b} \delta^a_{c]} - e_{bcd} \gamma^{da}. \tag{2.4}$$

It follows from (1.6) and (2.4) that

$$\begin{aligned} [\boldsymbol{E}_3, \boldsymbol{E}_2] &= \gamma^{11} \boldsymbol{E}_1 + \left(\gamma^{12} + \frac{1}{2} t_3 \right) \boldsymbol{E}_2 + \left(\gamma^{13} - \frac{1}{2} t_2 \right) \boldsymbol{E}_3, \\ [\boldsymbol{E}_1, \boldsymbol{E}_3] &= \left(\gamma^{21} - \frac{1}{2} t_3 \right) \boldsymbol{E}_1 + \gamma^{22} \boldsymbol{E}_2 + \left(\gamma^{23} + \frac{1}{2} t_1 \right) \boldsymbol{E}_3, \\ [\boldsymbol{E}_2, \boldsymbol{E}_1] &= \left(\gamma^{31} + \frac{1}{2} t_2 \right) \boldsymbol{E}_1 + \left(\gamma^{32} - \frac{1}{2} t_1 \right) \boldsymbol{E}_2 + \gamma^{33} \boldsymbol{E}_3. \end{aligned} \tag{2.5}$$

Let ∇^g denote the Levi-Civita covariant derivative based on the intrinsic metric tensor $g[\Phi]$ of (1.9) (Choquet-Bruhat *et al.*, 1977). Then

$$\begin{aligned} \nabla^g \boldsymbol{E}_a &= \omega_a{}^b \otimes \boldsymbol{E}_b, \\ \omega_a{}^b &= \omega_c{}^b{}_a E^c, \quad \omega^a_{[ab]} = \frac{1}{2} c^a_{bc}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} div_g \boldsymbol{E}_a &= \nabla^g_A \underset{a}{e}{}^A = g^{-1/2} \partial_A \left(g^{1/2} \underset{a}{e}{}^A \right), \\ g &= \det(g_{AB}), \end{aligned} \tag{2.7}$$

where (1.3) and (1.9) were taken into account. It follows from (1.3), (1.4), (2.4), (2.6), and (2.7) that the following *self-balance equations* would be fulfilled:

$$\mathrm{div}_g \boldsymbol{E}_a = t_a, \quad a = 1, 2, 3. \tag{2.8}$$

We see that the long-range distortion of a Bravais crystal with many dislocations can be characterized, in a continuous limit, by a pair $(\gamma, \boldsymbol{t})$ where

$$\begin{aligned} \gamma &= \gamma^{ab}\boldsymbol{E}_a \otimes \boldsymbol{E}_b, \quad \gamma^{ab} = \gamma^{ba};\ [\gamma] = \mathrm{cm}^{-3}, \\ \boldsymbol{t} &= {}^{a}\boldsymbol{E}_a, \quad t^a = \delta^{ab}t_b;\ [t] = \mathrm{cm}^{-2}. \end{aligned} \tag{2.9}$$

Note that if rank $\gamma \geq 2$, then the system $\Psi = (l_a; a = 1, 2, 3)$ of its principal vectors is defined univocally up to its orientation. Therefore, we can assume that

$$\begin{aligned} \gamma &= \gamma^a \boldsymbol{l}_a \otimes \boldsymbol{l}_a, \\ \boldsymbol{l}_a \cdot \boldsymbol{l}_b &= \delta_{ab}, \qquad \gamma^a \in C^\infty(\mathcal{B}), \\ [\boldsymbol{l}_a] &= [\gamma^a] = \mathrm{cm}^{-1}, \end{aligned} \tag{2.10}$$

where, according to (1.9) and (2.10)

$$\begin{aligned} \boldsymbol{l} &= Q_a{}^b\boldsymbol{E}_b, \quad Q_b{}^a \in C^\infty(\mathcal{B}), \\ Q &= (Q_b{}^a) : \mathcal{B} \to SO(E_3). \end{aligned} \tag{2.11}$$

Let $L_d(\mathcal{B})$ denote a finite total length of dislocation lines contained in the Bravais crystal $\mathcal{B} \subset E^3$. The *scalar volume dislocation density* ρ is defined by the condition that (Trzęsowski, 2001)

$$\begin{aligned} 0 \langle L_d(\mathcal{B}) &= \int_B \rho dV_g \langle \infty, \\ \rho &\in C^\infty(\mathcal{B}), \quad \rho\rangle 0,\ [\rho] = \mathrm{cm}^{-2} \end{aligned} \tag{2.12}$$

where dV_g denotes the Riemannian volume element (Eisenhart, 1964). The pair (ρ, α) of dislocation densities defines the so-called local Burgers vector $\boldsymbol{b}$ of a congruence of effective Volterra-type dislocation lines with their tangent $\boldsymbol{g}[\boldsymbol{\Phi}]$-versor $\boldsymbol{l}$ (Trzęsowski, 1994, 2000, 2001)

$$\begin{aligned} &\rho\boldsymbol{b} = \boldsymbol{l}\boldsymbol{\alpha}, \\ &b_g = \|\boldsymbol{b}\|_g = (\boldsymbol{b} \cdot \boldsymbol{b})^{1/2}\rangle 0, \quad \|\boldsymbol{l}\|_g = 1, \\ &[\boldsymbol{l}] = \mathrm{cm}^{-1}, \quad [\boldsymbol{b}] = [1], \quad [b_g] = \mathrm{cm}. \end{aligned} \tag{2.13}$$

There exist types of effective dislocations defined by relationships between the tangent vector $\boldsymbol{l}$ and the local Burgers vector $\boldsymbol{b}$: edge, screw, and mixed (Trzęsowski, 2001). If the local Burgers vector of (2.13) vanish then the corresponding congruence of curves does not consist of dislocation lines. If $\boldsymbol{l}$ is a principal vector of γ

$$\gamma\boldsymbol{l} = \eta\boldsymbol{l}, \quad \|\boldsymbol{l}\|_g = 1, \tag{2.14}$$

and the $\boldsymbol{l}$-congruence is endowed with a nonvanishing local Burgers vector, then this congruence of dislocations and its local Burgers vector are called *principal*

(Trzęsowski, 2000, 2001). If rank $\gamma \geq 2$, then there exist three distinguished congruences of curves and at least two of them are principal congruences of dislocations (edge, screw, or mixed) (Trzęsowski, 2001). Particularly, it follows from (2.1), (2.2), (2.8)–(2.10), and (2.13) that these are principal congruences of screw dislocations if and only if

$$\mathrm{div}_g \boldsymbol{E}_a = 0, \quad a = 1, 2, 3. \tag{2.15}$$

Note that if $\gamma = 0$, then any $\boldsymbol{l}$-congruence, except the case $l = t/\|t\|_g$ for which $b_g = 0$, consists of edge dislocations (Trzęsowski, 2001).

3. JACOBI IDENTITY

Let $W(\mathcal{B})$ denote the set of all smooth vector fields on $\mathcal{B}$ tangent to $\mathcal{B}$ and identified with the linear first-order differential operators $\boldsymbol{u} = u^A\partial_A$, $u^A \in C^\infty(\mathcal{B})$, where $\boldsymbol{u}(p) = u^A(X(p))\partial_{A|p}$ (Section 1). Since $W(\mathcal{B})$ is closed under addition of these vector fields and their multiplication by smooth functions on $\mathcal{B}$, it is the so-called *linear module* (Choquet-Bruhat *et al.*, 1977; Sikorski, 1972). A Bravais moving $\Phi = (\boldsymbol{E}_a)$ of (1.3) is a *vectorial base* of $W(\mathcal{B})$. It means that a vector field $\boldsymbol{u} \in W(\mathcal{B})$ can be univocally represented in the form $\boldsymbol{u} = u^a\boldsymbol{E}_a$, $u^a \in C^\infty(\mathcal{B})$ (Sikorski, 1972). We can also define an internal operation in $W(\mathcal{B})$ by means of the Lie bracket (see 1.6)

$$[\boldsymbol{u}, \boldsymbol{v}] = \boldsymbol{u} \circ \boldsymbol{v} - \boldsymbol{v} \circ \boldsymbol{u}. \tag{3.1}$$

The multiplication defined by the Lie bracket is distributive with respect to addition and anticommutative; it is not associative but it satisfies instead the *Jacobi identity*.

$$[\boldsymbol{u}_1, [\boldsymbol{u}_2, \boldsymbol{u}_3]] + [\boldsymbol{u}_2, [\boldsymbol{u}_3, \boldsymbol{u}_1]] + [\boldsymbol{u}_3, [\boldsymbol{u}_2, \boldsymbol{u}_1]] = 0. \tag{3.2}$$

The module $W(\mathcal{B})$ together with the above-defined internal operation is a *Lie algebra* (Choquet-Bruhat *et al.*, 1977).

It follows from (1.6) and (2.4) that applying the Jacobi identity to three vector fields $\boldsymbol{E}_a$, $\boldsymbol{E}_b$, and $\boldsymbol{E}_c$, the following *self-balance equation* should be fulfilled:

$$\partial_b\alpha^{ab} = -\alpha^{ab}t_b, \tag{3.3}$$

where ∂_a denotes the derivative in the direction $e_a = \underset{a}{e}{}^A C_A$, that is (see (1.2) and (1.3)):

$$\partial_a = \partial_{e_a} = \underset{a}{e}{}^A\partial_A = E_a, \tag{3.4}$$

or, equivalently

$$\partial_b\gamma^{ab} + \frac{1}{2}e^{abc}\partial_b t_c = -\gamma^{ab}t_b, \tag{3.5}$$

where (2.1) and (2.2) were taken into account. Particularly, in the case (2.15) we obtain that

$$\partial_b \gamma^{ab} = 0. \tag{3.6}$$

Let us consider the case

$$\begin{aligned} \gamma^{ab} t_b &= 0, \\ t_g &= \|t\|_g = (t_a t^a)^{1/2} \geq 0. \end{aligned} \tag{3.7}$$

Then

$$\partial_b \alpha^{ab} = \partial_b \gamma^{ab} + \frac{1}{2} e^{abc} \partial_b t_c = 0. \tag{3.8}$$

Moreover, according to (2.10) and (2.11), we can take without loss of generality that

$$t = t^3 l_3, \quad l_3 = \boldsymbol{E}_3, \tag{3.9}$$

and

$$\begin{aligned} Q_b{}^a &= \cos\theta \delta_b^a + (1 - \cos\theta)\delta_3^a \delta_{3b} - \sin\theta \in_{bc}{}^a, \\ \in_{bc}{}^a &= \delta^{ad} \in_{dbc}, \quad \theta : \mathcal{B} \to \langle 0, \pi \rangle. \end{aligned} \tag{3.10}$$

Then

$$\begin{aligned} l_1 &= \cos\theta \boldsymbol{E}_1 + \sin\theta \boldsymbol{E}_2, \\ l_2 &= -\sin\theta \boldsymbol{E}_1 + \cos\theta \boldsymbol{E}_2. \end{aligned} \tag{3.11}$$

It follows from (2.1), (2.2), (2.9), (2.10), (2.13), (2.14), (3.7), and (3.9) that the three principal congruences of dislocations defined by $g[\boldsymbol{\Phi}]$-versors $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, and $\boldsymbol{l}_3$ are endowed with the principal local Burgers vectors (Section 2) $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ given by

$$\begin{aligned} \rho b_1 &= \gamma^1 l_1 + \mu m_1, \\ \rho b_2 &= \gamma^2 l_2 + \mu m_2, \\ \rho b_3 &= \gamma^3 l_3, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} &\mu = t_g/2, \quad t_g = |t^3|, \quad \gamma^3 \mu = 0, \\ &m_1 = -l_2, \quad m_2 = l_1, \quad l_a \cdot l_b = \delta_{ab}. \end{aligned} \tag{3.13}$$

Thus

$$\begin{aligned} \rho b_{g,\alpha} &= \sqrt{(\gamma^\alpha)^2 + \mu^2}, \quad \alpha = 1, 2, \\ \rho b_{g,3} &= |\gamma^3|, \quad b_{g,a} = \|b_a\|_g. \end{aligned} \tag{3.14}$$

and

$$\cos\psi = \frac{b_1 \cdot b_2}{b_{g,1} b_{g,2}} = 2\mu(\gamma^1 - \gamma^2)/b_{g,1} b_{g,2}. \tag{3.15}$$

It follows that

$$\psi = \frac{\pi}{2} \quad \text{iff} \quad \mu(\gamma^1 - \gamma^2) = 0. \tag{3.16}$$

If

$$\gamma^3 = 0, \quad t_g \geq 0, \tag{3.17}$$

then $b_3 = 0$, which means that the principal vector $\boldsymbol{l}_3$ does not define a congruence of dislocations. The two remaining principal vectors $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ define congruences of mixed dislocations with local Burgers vectors b_1 and b_2 of the following form:

$$\begin{aligned} \rho b_1 &= (\gamma^1 \cos\theta + \mu \sin\theta)\boldsymbol{E}_1 + (\gamma^1 \sin\theta - \mu\cos\theta)\boldsymbol{E}_2, \\ \rho b_2 &= (-\gamma^2 \sin\theta + \mu\cos\theta)\boldsymbol{E}_1 + (\gamma^2 \cos\theta + \mu\sin\theta)\boldsymbol{E}_2, \end{aligned} \tag{3.18}$$

where (3.11)–(3.13) were taken into account. Thus, we are dealing with such principal congruences of dislocations that their tangents ($\boldsymbol{l}_1$ and $\boldsymbol{l}_2$) and their local Burgers vectors (b_1 and b_2) are located on the same planes of the two-dimensional distribution $\pi(E_1, E_2)$ of local crystal planes spanned, at each point $p \in \mathcal{B}$, by vectors $E_1(p)$ and $E_2(p)$. It means that $\pi(\boldsymbol{E}_1, \boldsymbol{E}_2)$-planes are virtually *local slip planes* (Trzęsowski, 2001). If additionally, (3.6) is fulfilled, then

$$dt = 0, \quad \text{i.e. } t = t_3 E^3 = d\varphi, \tag{3.19}$$

where (3.8) and (3.9) were taken into account, and the crystal surfaces (Section 1)

$$\textstyle\sum_c = \varphi^{-1}(c), \quad c \in R, \tag{3.20}$$

considered as surfaces in the material space $\mathcal{B}_g$, define in this space a representation of *virtual slip surfaces* normal to $\boldsymbol{E}_3$-direction.

The principal Burgers vectors b_1 and b_2 of (3.18) are tangent to lattice lines, analogically as it takes place in the case of single dislocations in discrete crystal structures (Hull and Bacon, 1984), only if

$$\begin{aligned} &\gamma^1 = \gamma^2 = \gamma; \mathcal{B} \to (0, \infty), \quad \gamma^3 - 0, \\ &\mu = |\kappa| = \gamma tg\theta, \quad \kappa = \frac{1}{2} t_3, \quad \theta : \mathcal{B} \to \langle 0, \pi/2 \rangle, \end{aligned} \tag{3.21}$$

where $\gamma, \theta \in C^\infty(\mathcal{B})$. Then

$$\begin{aligned} &\rho b_\alpha = (\gamma/\cos\theta)\boldsymbol{E}_\alpha, \quad \alpha = 1, 2, \\ &b_1 \cdot b_2 = 0, \quad b_{g,1} = b_{g,2} = b_g, \end{aligned} \tag{3.22}$$

where

$$\rho b_g - \sqrt{\gamma^2 + \kappa^2} = \gamma / \cos\theta. \tag{3.23}$$

Particularly, it follows from (2.5), (2.10), (3.9), (3.11), and (3.21) that for $\theta = 0$, we have

$$t_g = 0, \quad l_a = \boldsymbol{E}_a, \ \ a = 1, 2, 3, \tag{3.24}$$

and

$$[\boldsymbol{E}_2, \boldsymbol{E}_1] = 0, \quad [\boldsymbol{E}_1, \boldsymbol{E}_3] = \gamma \boldsymbol{E}_2, \quad [\boldsymbol{E}_3, \boldsymbol{E}_2] = \gamma \boldsymbol{E}_1, \tag{3.25}$$

where the Bravais moving frame $\Phi = (\boldsymbol{E}_a)$ is additionally constrained by (2.15). It follows from (3.22)–(3.24) that we are dealing with two principal congruences consisting of screw dislocations and such that

$$\rho b_g = \gamma. \tag{3.26}$$

If (see the remark at the very end of Section 2)

$$\begin{aligned} &\gamma = 0, \quad t = t^3 \boldsymbol{E}_3, \ \ t_3 \rangle 0, \\ &l_\alpha = E_\alpha, \quad \alpha = 1, 2, \end{aligned} \tag{3.27}$$

then (2.5) reduces to

$$\begin{aligned} &[\boldsymbol{E}_1, \boldsymbol{E}_2] = 0, \quad [\boldsymbol{E}_3 . \boldsymbol{E}_\alpha] = \kappa \boldsymbol{E}_\alpha, \quad \alpha = 1, 2, \\ &\kappa = \frac{1}{2} t_3 : \mathcal{B} \to (0, \infty), \end{aligned} \tag{3.28}$$

and, according to (3.12) and (3.13), we have

$$\rho b_1 = \kappa \boldsymbol{E}_2, \quad \rho b_2 = \kappa \boldsymbol{E}_1, \ \ b_3 = 0, \tag{3.29}$$

so, that $b_{g,1} = b_{g,2} = b_g$ where

$$\rho b_g = \kappa. \tag{3.30}$$

It means that the E_α-congruences, $\alpha = 1, 2$, consist of edge dislocations of the same strength.

If additionally there exists a smooth function $H{:}R \to (0, \infty)$, $[H] = \mathrm{cm}^{-1}$, such that

$$\forall p \in \sum\nolimits_c, \quad \kappa(p) = H(c), \tag{3.31}$$

where $\sum_c \subset \mathcal{B}_g$ denotes the crystal surfaces defined by (3.20) and considered as a two-dimensional submanifold of the material space $\mathcal{B}_g$, then $\sum_c$ is a flat surface with the constant mean curvature $H(c)$ (Trzęsowski, 2001). Thus, it is virtually a *single glide* case (Trzęsowski, 2001).

4. BIANCHI-TYPE DISTORTIONS

A basis $\Phi = (\boldsymbol{E}_a)$ of the linear module $W(\mathcal{B})$ (Section 3) spans a three-dimensional real vector space $W_\Phi(\mathcal{B})$ of the so-called Φ-parallel vector fields on $\mathcal{B}$

$$W_\Phi(\mathcal{B}) = \{v = v^a \boldsymbol{E}_a, v^a \in R\} \subset W(\mathcal{B}). \tag{4.1}$$

If the object of anholonomity of (1.6) consists of constants

$$C^a_{bc} = \text{const.}, \tag{4.2}$$

then the Lie bracket of (3.1) defines in the vector space $W_\Phi(\mathcal{B})$ the structure of a three-dimensional real Lie algebra. The components of geometric objects appearing in (2.1)–(2.5) are constants and (3.5) reduces to (3.7). Therefore, (2.11) and (3.9)–(3.11) with $Q_b{}^a = \text{const.}$ would be valid and, without loss of generality, we can restrict ourselves to the case

$$l_a = E_a, \quad a = 1, 2, 3. \tag{4.3}$$

The commutation rules of (2.5) takes, according to (2.10) and (3.9), the following canonical form (Dubrovin *et al.*, 1979):

$$\begin{aligned} &[\boldsymbol{E}_3, \boldsymbol{E}_2] = \gamma^1 \boldsymbol{E}_1 + \kappa \boldsymbol{E}_2, \quad [\boldsymbol{E}_1, \boldsymbol{E}_3] = -\kappa \boldsymbol{E}_1 + \gamma^2 \boldsymbol{E}_2, \\ &[\boldsymbol{E}_2, \boldsymbol{E}_1] = \gamma^3 \boldsymbol{E}_3, \end{aligned} \tag{4.4}$$

where

$$\kappa = \frac{1}{3} t_3, \quad \kappa \gamma^3 = 0. \tag{4.5}$$

Moreover, it follows from (2.8), (3.9), and (4.5) that the following conditions would be fulfilled

$$\text{div}_g \boldsymbol{E}_1 = \text{div}_g \boldsymbol{E}_2 = 0, \quad \text{div}_g \boldsymbol{E}_3 = 2\kappa. \tag{4.6}$$

The principal vectors of (2.10) and (4.3) define principal congruences of dislocations endowed with the (principal) local Burgers vectors b_a, $a = 1, 2, 3$, given by

$$\begin{aligned} \rho b_1 &= \gamma^1 \boldsymbol{E}_1 - \mu \boldsymbol{E}_2, \\ \rho b_2 &= \gamma^2 \boldsymbol{E}_2 - \mu \boldsymbol{E}_1, \\ \rho b_3 &= \gamma^3 \boldsymbol{E}_3; \quad \mu = |\kappa| \geq 0. \end{aligned} \tag{4.7}$$

If $\kappa \neq 0$, then $\gamma^3 = 0$ what means that $\boldsymbol{E}_3$ does not define a congruence of dislocations; $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$-congruences consist of mixed or edge dislocations. If $\gamma^3 \neq 0$, then $\mu = 0$ what means that we are dealing with congruences of screw dislocations constrained by (2.15). It follows from (4.7) that

$$\rho b_{g,a} = \text{const.}, \quad a = 1, 2, 3, \tag{4.8}$$

where $b_{g,a} = \|b_a\|_g$, and thus

$$p = \text{const.}, \quad \text{iff} \quad b_{g,a} = \text{const.}, \quad a = 1, 2, 3. \tag{4.9}$$

We will call *uniformly dense* such distributions of dislocations for which

$$\alpha^{ab} = \text{const.}, \quad \rho = \text{const.}, \tag{4.10}$$

It ought to be stressed that the scalar volume dislocation density ρ of a finite total length $L_d(\mathcal{B})$ of dislocations contained in the Bravais crystal $\mathcal{B}$ (Section 2) is measured with respect to material volume element dV_g. Consequently, for uniformly dense distributions of dislocations, we have:

$$\begin{aligned} &\rho = L_d(\mathcal{B})/V_g(\mathcal{B}), \\ &0 \ \langle \ V_g(\mathcal{B}) = \int_{\mathcal{B}} dV_g \langle \infty, \end{aligned} \tag{4.11}$$

where (2.12) and (4.7) were taken into account.

The Bianchi classification of three-dimensional real Lie algebras (Dubrovin *et al.*, 1979) offers an opportunity for a comprehensive classification of all uniformly dense distributions of dislocations. This classification can be just as well interpreted as a list of *elementary Bianchi-type distortions* of continuized Bravais crystal due to dislocations (see Section 5—Final remarks).

Let us consider, as an example, the case of mutually orthogonal principal local Burgers vectors. It follows from (3.16) and (4.7) that then the following conditions would be fulfilled

$$\kappa(\gamma^1 - \gamma^2) = 0, \quad \kappa\gamma^3 = 0. \tag{4.12}$$

Particularly, if

$$\kappa \neq 0, \tag{4.13}$$

then

$$\gamma^1 = \gamma^2 = \gamma, \quad \gamma^3 = 0. \tag{4.14}$$

If

$$\gamma = 0, \quad \kappa\rangle 0, \tag{4.15}$$

then, according to (4.4), the commutation rules of (3.28) with $\kappa = \text{const.}$ take place. It corresponds to a distribution of edge dislocations (Trzęsowski, 2001) of Bianchi-type V. The material space $\mathcal{B}_g$ is then a particular case of the so-called equidistant Riemmanian space (Trzęsowski, 2001). Moreover, in this case the material space has a constant negative scalar curvature

$$K = -\kappa^2, \tag{4.16}$$

and the flat virtual slip surfaces $\sum_c$, $c \in R$, of (3.20) has the same constant mean curvature H. It follows from (3.30) and (3.31) that for these slip surfaces

$$H = \rho b_g. \tag{4.17}$$

Note that if ρ is interpreted as the mean density of dislocations in a Bravais crystal and b_g is identified with the mean strength of these dislocations, then (4.17) covers with the well-known approximation of the normal curvature H of crystal surfaces in their local crystallographic directions (Trzęsowski, 2001).

If

$$\gamma \rangle 0, \quad \kappa \rangle 0, \tag{4.18}$$

then (4.4) can be written in the form

$$\begin{aligned} &[\boldsymbol{E}_3, \boldsymbol{E}_2] = \gamma[\boldsymbol{E}_1 + \zeta \boldsymbol{E}_2], \quad [\boldsymbol{E}_1, \boldsymbol{E}_3] = \gamma[\boldsymbol{E}_1 - \zeta \boldsymbol{E}_2], \\ &[\boldsymbol{E}_2, \boldsymbol{E}_1] = 0, \end{aligned} \tag{4.19}$$

where ζ is the parameter describing, according to (4.7), (4.14), and (4.18), the ratio of edge and screw components of the principal local Burgers vectors b_1 and b_2

$$\zeta = \frac{\kappa}{\gamma} \rangle 0. \tag{4.20}$$

The commutation rules of (4.19) define the Bianchi Lie algebra of type VII(ζ), $\zeta \rangle 0$. The such defined distributions of dislocations have principal congruences of mixed dislocations of the same strength given by

$$\begin{aligned} b_{g,2} = b_{g,1} &= b_g(\zeta), \\ \rho b_g(\zeta) &= \gamma\sqrt{1+\zeta^2}. \end{aligned} \tag{4.21}$$

Note that if $b_g(\zeta_1) \neq b_g(\zeta_2)$, then the corresponding Lie algebras VII(ζ) and VII(ζ) are not isomorphic (Dubrovin *et al.*, 1979). Thus, we are dealing with a one parameter family of physically different distortions of a Bravais crystal structure due to many dislocations.

If

$$\gamma_1 = \gamma_2 = \gamma \rangle 0, \quad \zeta \to 0_+, \tag{4.22}$$

then VII(ζ) tends to the Bianchi Lie algebra VII(0) defined by (3.25) with $\gamma =$ const. and the material space $\mathcal{B}_g$ is flat (Trzęsowski, 2001). It means that the secondary point defects have no influence on metric properties of the dislocated Bravais crystal. Thus, in this case (2.15) is fulfilled for a flat intrinsic metric $g[\Phi]$.

5. FINAL REMARKS

Let us return to the tensor measure $\boldsymbol{S}[\Phi]$ of anholonomity of a Bravais moving frame $\Phi = (E_a)$ (Section 1). It is easy to see that the dependence $\Phi \to \boldsymbol{S}[\Phi]$ is invariant under the group $GL^+(3)$ of all real 3×3 matrices with positive determinant (Trzęsowski and Sławianowski, 1990)

$$\forall \boldsymbol{L} \in GL^+(3), \quad \boldsymbol{S}[\Phi \boldsymbol{L}] = \boldsymbol{S}[\Phi],$$

$$\Phi \boldsymbol{L} = (\boldsymbol{E}_a L_b{}^a; b = 1, 2, 3), \quad \boldsymbol{L} = (L_b{}^a;\ a, b = 1, 2, 3). \tag{5.1}$$

This global invariance means that homogeneous deformations of a body with dislocations do not influence the fundamental physical field describing the long-range distortion of the crystalline structure because of dislocations. If we restrict ourselves to unimodular homogeneous deformations (i.e. $\boldsymbol{L} \in SL(3)$), then the scalar volume density is additionally preserved.

Therefore, we can anticipate the same global invariance of Euler–Lagrange equations describing a static and self-equilibrium distribution of dislocations. It can be shown that there exists such a class of affinely invariant Euler–Lagrange equations that uniformly dense distributions of dislocations of orthogonal types (i.e., so(3) or so(2,1)-type) are described by universal solutions of these equations (Trzęsowski and Sławianowski, 1990). It suggests that the uniformly dense distribution of dislocations can be treated as a *fundamental state* of the distorted Bravais structure. Therefore, a fundamental state here is not the state of an ideal crystal structure, but the state of its elementary distortion due to dislocations.

REFERENCES

Choquet-Bruhat, Y., DeWitt-Morette, C., and Dillard-Bleick, M. (1977). *Analysis, Manifolds and Physics*, North-Holland, Amsterdam.

Dubrovin, B. A., Novikov, S. P., and Fomenko, A. T. (1979). *Modern Geometry*, "Nauka," Moscow. [in Russian]

Eisenhart, L. P. (1964). *Riemannian Geometry*, Princeton University Press, Princeton.

Gołąb, S. (1966). *Tensor Calculus*, PWN, Warsaw. [in Polish]

Hull, D. and Bacon, D. J. (1984). *Introduction to Dislocations*, Pergamon, Oxford.

Kröner, E. (1986). *Zeitschrift fiir Angewandti Mathematical Mechanics* **66**, T284-T289.

Oding, I. A. (1961). *Theory of Dislocations in Metals*, PWT, Warsaw. [in Polish]

Orlov, A. N. (1983). *Introduction to the Theory of Defects in Crystals*, Vyscaja Skola, Moscow. [in Russian]

Sikorski, R. (1972). *Introduction to Differential Geometry*, PWN, Warsaw. [in Polish]

Trzęsowski, A. and Sławianowski, J. J. (1990). *International Journal of Theoretical Physics* **29**, 1239–1249.

Trzęsowski, A. (1993). *Reports on Mathematical Physics* **32**, 71–98.

Trzęsowski, A. (1994). *International Journal of Theoretical Physics* **33**, 931–950.

Trzęsowski, A. (2000). *Acta Mechanica* **141**, 173–192.

Trzęsowski, A. (2001). *International Journal of Theoretical Physics* **40**, 727–753.

Yano, K. (1958). *The Theory of Lie Derivatives and its Applications*, North-Holland, Amsterdam.